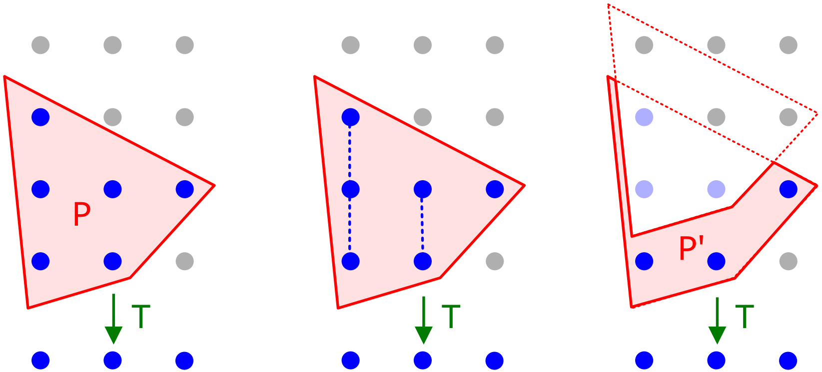
The Complexity of Presburger Arithmetic in Fixed Dimension

Kevin Woods
Oberlin College



Classic result

Theorem [Lenstra 1983]: Fix d. There is a polynomial time algorithm which, given a (rational) polyhedron $P \subseteq \mathbb{R}^d$ (input, e.g., as list of integral defining inequalities), decides if $P \cap \mathbb{Z}^d$ is nonempty.

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What next? How might we generalize?

Theorem [Barvinok 1994]: Fix d. There is a polynomial time algorithm which, given a polyhedron $P \subseteq \mathbb{R}^d$, counts $|P \cap \mathbb{Z}^d|$.

A key idea

Definition: For $S \subseteq \mathbb{Z}^d$, we can define the generating function

$$\sum_{(a_1, a_2, \ldots, a_d) \in S} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.$$

Example: $S = [5, 50] \cap \mathbb{Z}$ has generating function

$$x^5 + x^6 + \cdots + x^{50} = \frac{x^5 - x^{51}}{1 - x}.$$

Limit $x \to 1$ (with L'Hôpital's Rule) yields

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Moral:

- We can often use patterns in our sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets (like cardinality).

What next?

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Theorem [Barvinok–W 2003]: Fix d. There is a polynomial time algorithm which, given a polyhedron $P \subseteq \mathbb{R}^d$ and a linear transformation $T : \mathbb{Z}^d \to \mathbb{Z}^k$, computes the generating function for $T(P \cap \mathbb{Z}^d)$ (and hence can compute its cardinality).

Note: Most interesting when T has nontrivial kernel, e.g., some sort of projection.

Frobenius problem

Example: Given positive integers $a_1, \ldots, a_d$, let

$$P = \mathbb{R}^d_{\geq 0} \quad \text{and} \quad T(x_1, \ldots, x_d) = a_1 x_1 + \cdots + a_d x_d.$$

Then $S = T(P \cap \mathbb{Z}^d)$ is the set of nonnegative integer combinations of $a_1, \ldots, a_d$, that is, the semigroup generated by $a_1, \ldots, a_d$ (i.e., closure under addition).

- What is the largest integer not in S, assuming a_i relatively prime? (Frobenius problem)
- How many positive integers are not in S?
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- The largest integer not in S is $a_1 a_2 - a_1 - a_2$ [Sylvester 1884].
- The number of positive integers not in S is $(a_1 a_2 - a_1 - a_2 + 1)/2$.
- The generating function for S is

$$\sum_{n \in S} x^n = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}.$$

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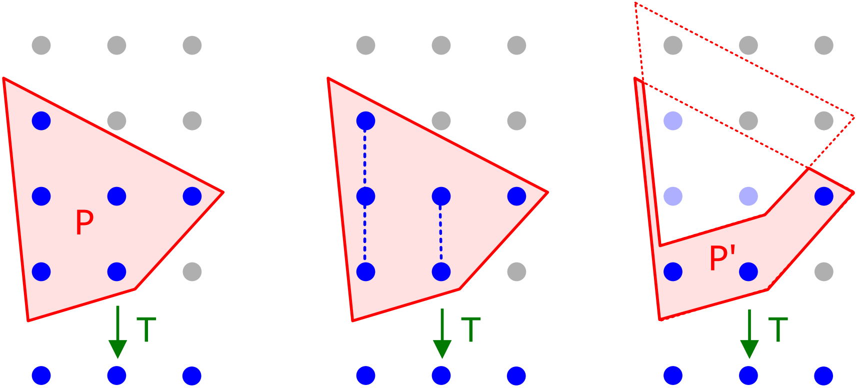
A key idea

A 1-dimensional kernel is pretty easy. E.g, $T(x, y) = x$:



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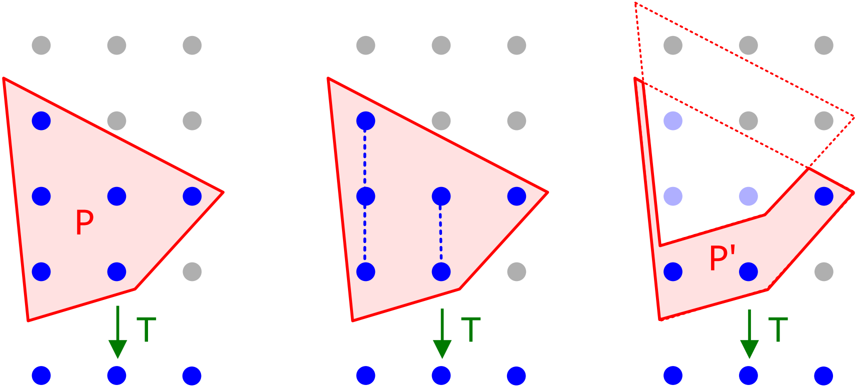

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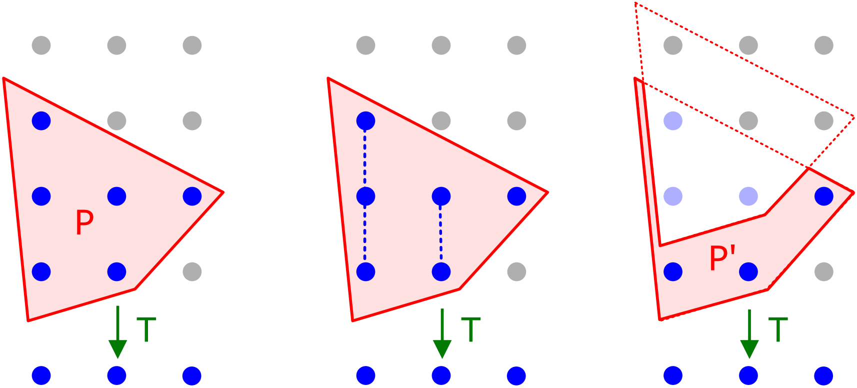

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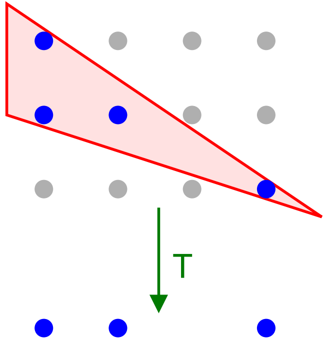
With a higher dimensional kernel, inductively project out one dimension at a time. But this may create gaps.



Key: Carefully control the gaps.

What next?

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If $T(x, y) = x$, then $T(P \cap \mathbb{Z}^2) = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z} \quad (x, y) \in P\}$.

Presburger Arithmetic: Sets defined over the integers using quantifiers and Boolean combinations ($\wedge$, $\vee$, $\neg$) of linear inequalities. So far, we've been using only conjunctions.

Theorem [Barvinok-W 2003]: Fix m, n, s. There is a polynomial time algorithm which, given a formula $\Phi(x, y)$ that is a Boolean combination of at most s linear inequalities in $x_1, \ldots, x_m$, $y_1, \ldots, y_n$, computes the generating function for

$$x \in \mathbb{Z}^m : \exists y \in \mathbb{Z}^n \; \Phi(x, y).$$

Similarly for $x \in \mathbb{Z}^m : \forall y \in \mathbb{Z}^n \; \Phi(x, y)$.

Note: If s is not fixed above, then the problem is NP-hard [Schöning 1997].

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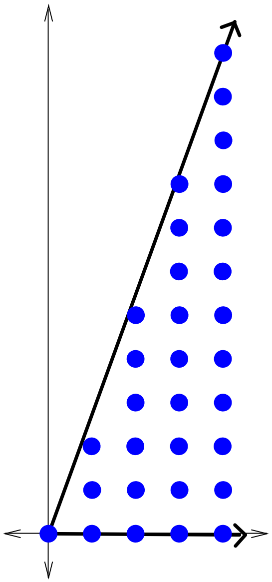
Hilbert bases



Given a (rational) cone $C \subseteq \mathbb{Z}^d$, $C \cap \mathbb{Z}^d$ is a semigroup (closed under addition).

Let S be the minimal set of generators (Hilbert Basis). In example, S is

$$(1,0),(1,1),(1,2),(2,5),(3,8),(4,11)$$

S is the set of $x \in \mathbb{Z}^d$ such that

$$\forall y,z \in \mathbb{Z}^d \ \big(y \in C \setminus \{0\} \ \wedge \ z \in C \setminus \{0\}\big) \Rightarrow x \neq y + z$$

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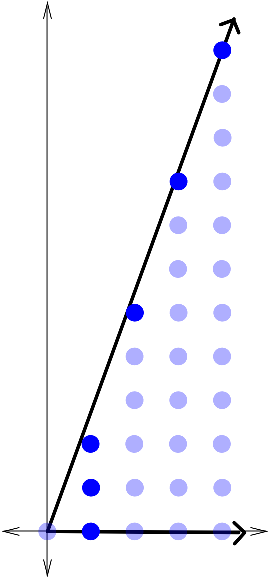

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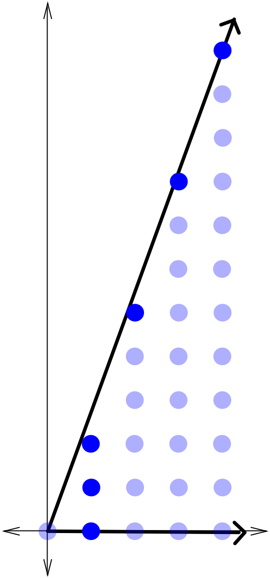

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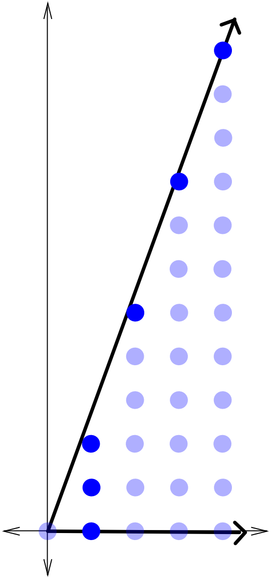

When $d = 2$ and cone has extreme rays (q, p) and $(1, 0)$, Hilbert basis is related to continued fraction expansion of p/q.

$$\frac{11}{4} = 2 + \frac{1}{1 + \frac{1}{3}}, \quad \frac{8}{3} = 2 + \frac{1}{1 + \frac{1}{2}}, \quad \frac{5}{2} = 2 + \frac{1}{1 + \frac{1}{1}},$$

$$\frac{3}{1} = 2 + \frac{1}{1} \text{ but } (1, 3) \notin C,$$

$$\frac{2}{1} = 2, \quad \frac{1}{1} = 1, \quad \frac{0}{1} = 0.$$

Note: the y-coordinates form non-overlapping arithmetic progressions: (0,1,2), (5,8,11).

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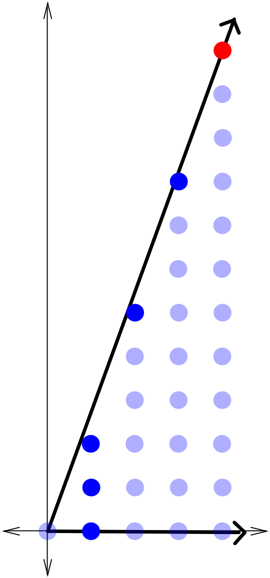

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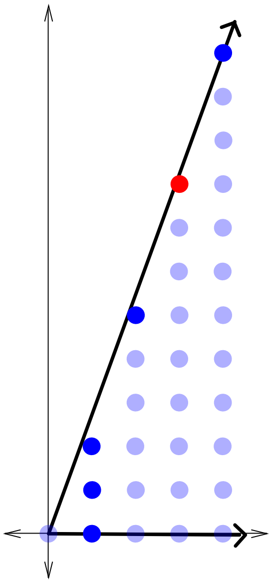

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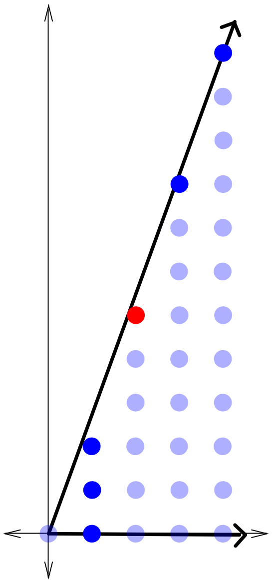

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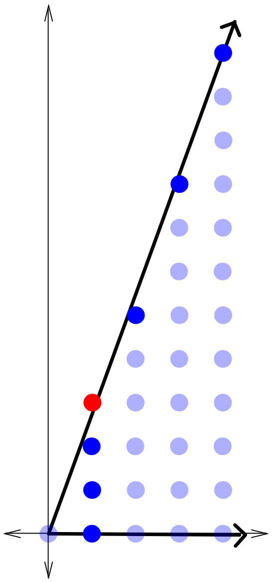

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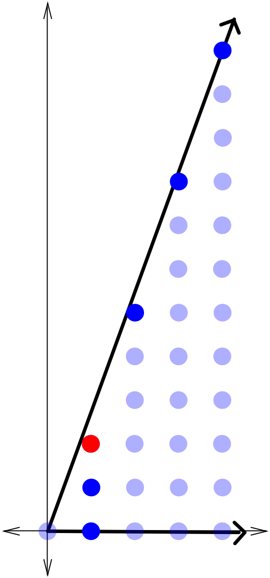

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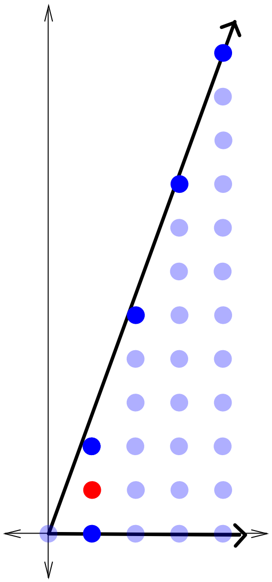
Hilbert bases



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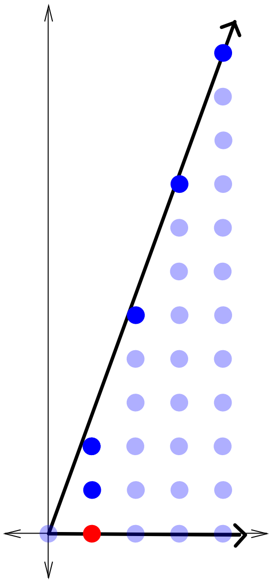

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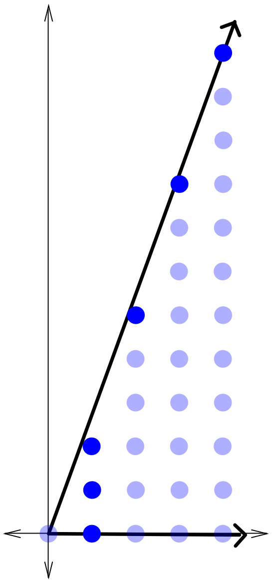

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Define the set of y coordinates of a particular Hilbert Basis, creating non-overlapping arithmetic progressions. Needs a $\forall$ quantifier.

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This looks like a known NP-complete problem: Given a set of arithmetic progressions and an interval, do the arithmetic progressions cover the interval?

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Thank You!