Presburger Arithmetic, Rational Generating Functions, and Quasi-polynomials

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Examples

Theme: Generating functions encode patterns of sets, in useful ways.

Definition: Given $S \subseteq \mathbb{N}^d$, define

$$f(S; x_1, x_2, \ldots, x_d) = \sum_{(a_1, a_2, \ldots, a_d) \in S} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.$$

Example: $S = \left\{ a \in \mathbb{N} : \ a \leq 5000 \right\}$. Then

$$f(S; x) = 1 + x + x^2 + \cdots + x^{5000}$$

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Example: $S = \{a \in \mathbb{N} : \ a \leq 5000\}$. Then

$$\begin{aligned} f(S; x) &= 1 + x + x^2 + \cdots + x^{5000} \\ &= \frac{1 - x^{5001}}{1 - x}. \end{aligned}$$

Examples

$$S = \left\{ a \in \mathbb{N} : \ \exists b \in \mathbb{N}, \ a = 2b+1, \ a \leq 5000 \right\}.$$

$$\begin{aligned} f(S;x) &= x + x^3 + x^5 + \cdots + x^{4999} \\ &= \frac{x - x^{5000}}{1-x^2}. \end{aligned}$$

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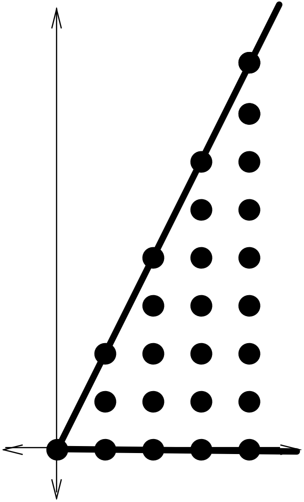

$$f(S;x,y) = 1 + x + xy + xy^2 + x^2 + \cdots$$

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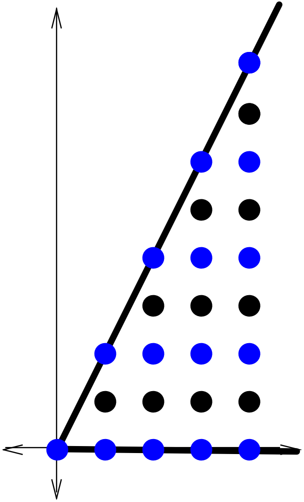

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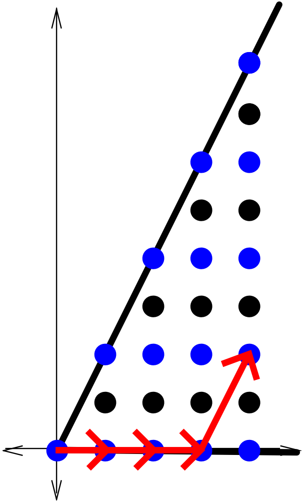

$$x^4 y^2 = (x)^3 (xy^2)^1$$

$$\begin{aligned} &(1 + x + x^2 + x^3 + \cdots) \\ &\cdot (1 + (xy^2)^1 + (xy^2)^2 + \cdots) \\ &= \frac{1}{(1-x)(1-xy^2)} \end{aligned}$$

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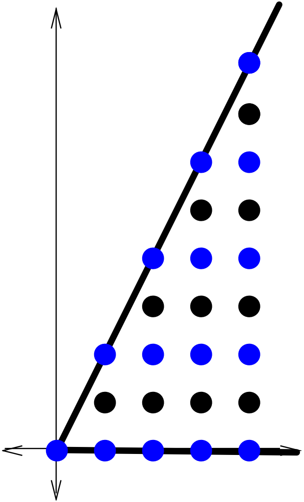

$$\begin{gathered}(1 + x + x^2 + x^3 + \cdots) \\ \cdot(1 + (xy^2)^1 + (xy^2)^2 + \cdots) \\ = \frac{1}{(1-x)(1-xy^2)}\end{gathered}$$

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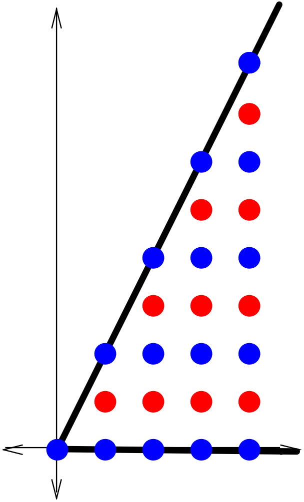

$$x^1 y^1$$
$$\cdot(1 + x + x^2 + x^3 + \cdots)$$
$$\cdot(1 + (xy)^1 + (xy)^2 + \cdots)$$
$$= \frac{xy}{(1-x)(1-xy^2)}$$

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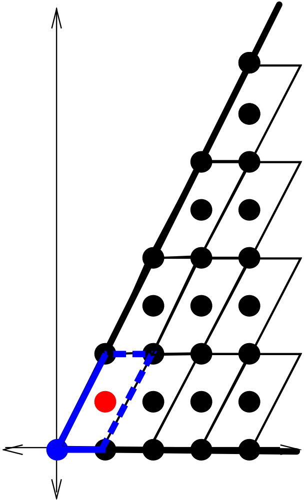

$$\frac{1 + xy}{(1-x)(1-xy^2)}$$

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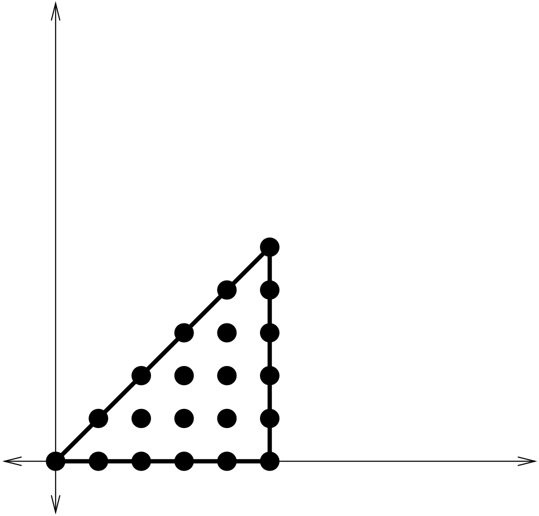
$$S = \left\{(a, b) \in \mathbb{N}^2 : \ b \leq a, \ a \leq 5\right\}$$



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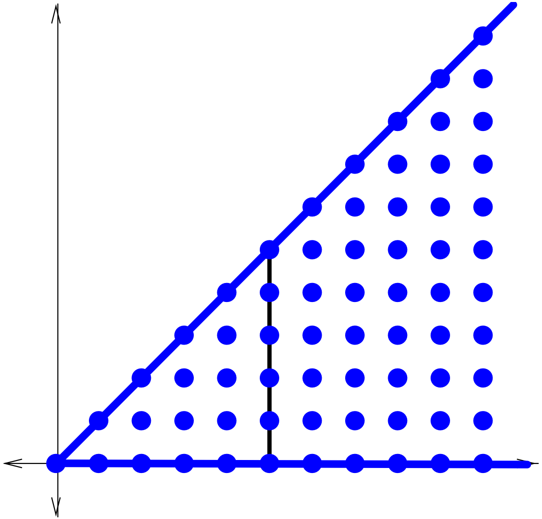

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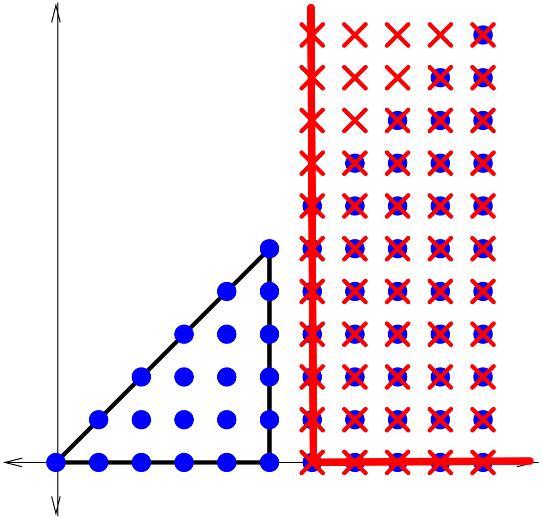

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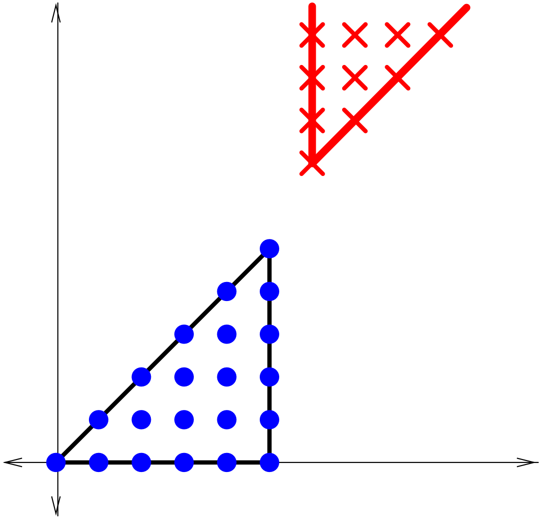

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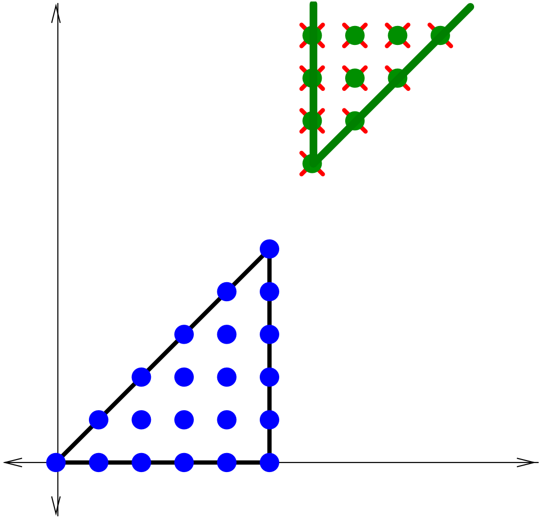

$$+\frac{x^6 y^7}{(1-xy)(1-y)}$$

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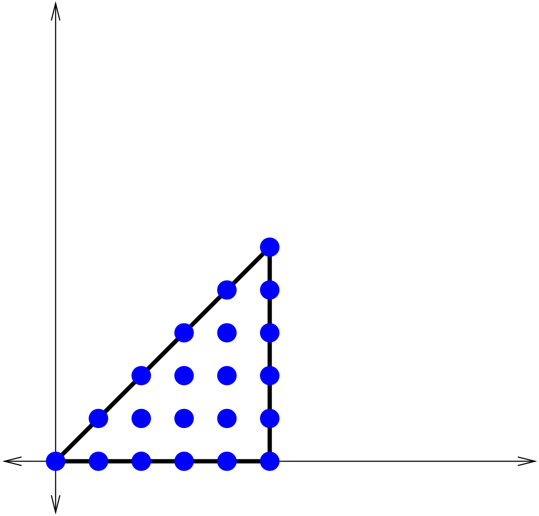

$$f(S;x,y) =$$

$$\frac{1}{(1-x)(1-xy)}$$

$$-\frac{x^6}{(1-x)(1-y)}$$

$$+\frac{x^6y^7}{(1-xy)(1-y)}.$$

Presburger Sets

Definition: A Presburger set is defined over $\mathbb{N}^d$ using quantifiers ($\exists$ and $\forall$), boolean operations (and, or, not), and linear (in)equalities ($\leq$, $=$, $>$).

Examples

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Presburger Sets

The generating function of a Presburger set is a rational function:

- Cones: see example (triangulate if not simplicial).
- Polyhedra: by inclusion-exclusion [Brion].
- Quantifier-free formulas: unions of polyhedra (DNF).
- All Presburger sets: quantifier elimination [Presburger].

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Presburger Sets

The following are equivalent:

- S is a Presburger set.
- $f(S; \mathbf{x})$ is a rational generating function.
- S is a finite union of sets of the form $P \cap (\lambda + \Lambda)$, where P is a polyhedron, $\lambda \in \mathbb{N}^d$, and $\Lambda \subseteq \mathbb{Z}^d$ is a lattice.
  [cf. semi-linear sets of Ginsburg, Spanier]

The Power of Generating Functions

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The generating function contains all of the information of the set, in a way that can be exploited.

$$
\begin{aligned}
f(S;1) &= |S|. \\
\frac{\partial}{\partial x_1} f(S;\mathbf{x})\Big|_{\mathbf{x}=1} &= \sum_{\mathbf{a}\in S} a_1. \\
\text{degree } f\left(S; z^{c_1},\ldots,z^{c_d}\right) &= \max_{\mathbf{a}\in S} \mathbf{c}\cdot\mathbf{a}.
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For fixed dimension, given rational generating functions $f(S;\mathbf{x})$ and $f(T;\mathbf{x})$, there are polynomial time algorithms to compute

- $f(S;1)$
- $\frac{\partial}{\partial x_1} f(S;\mathbf{x})$
- degree $f\left(S; z^{c_1}, \ldots, z^{c_d}\right)$
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[Barvinok, W],

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Proofs using generating functions:

- For fixed dimension, the number of solutions to a quantifier-free Presburger formula (e.g., a polyhedron) is computable in polynomial time. [Barvinok]
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Open Problem: What if there is quantifier alternation? Don't even know that the existence of solutions can be decided in polynomial time.

Parametric Counting

$$S_t = \{a \in \mathbb{N} : \ 2a \le t\}$$

Then

$$\begin{aligned} g(t) &\doteq |S_t| \\ &= \left\lfloor \frac{t}{2} \right\rfloor + 1 \\ &= \begin{cases} \frac{t+2}{2} & \text{if } t \equiv 0 \bmod 2, \\ \frac{t+1}{2} & \text{if } t \equiv 1 \bmod 2 \end{cases} \end{aligned}$$

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\begin{aligned}
\sum_t g(t)x^t &= \sum_t \left(\left\lfloor \frac{t}{2} \right\rfloor + 1 \right) x^t \\
&= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots \\
&= (1 + x)(1 + 2x^2 + 3x^4 + \cdots) \\
&= \frac{1 + x}{(1 - x^2)^2},
\end{aligned}
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by substituting $y = x^2$ into

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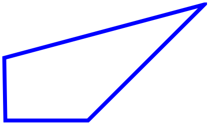
This is a rational generating function!!!

Parametric Counting

With more than 1 parameter, need piecewise quasi-polynomials.

Example: $S_{s,t} = \{a, b \in \mathbb{N} : \ 2b - a \leq 2t - s, \ a - b \leq s - t\}$



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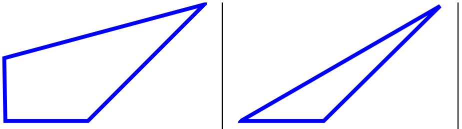

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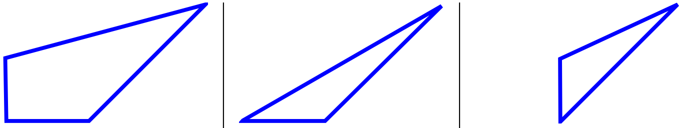

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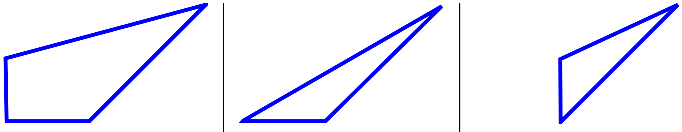

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$$|S_{s,t}| = \begin{cases} \frac{s^2}{2} - \lfloor \frac{s}{2} \rfloor s + \frac{s}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } t \leq s \leq 2t \\ st - \lfloor \frac{s}{2} \rfloor s - \frac{t^2}{2} + \frac{t}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } 0 \leq 2t \leq s \\ \frac{t^2}{2} + \frac{3t}{2} + 1 & \text{if } 0 \leq s \leq t \end{cases}.$$

Parametric Counting

Given a function $g : \mathbb{N}^n \to \mathbb{Q}$ and the following three possible properties:

A. g parametrically counts solutions to a Presburger formula,

B. g is a piecewise quasi-polynomial, and

C. $\sum_{\mathbf{p} \in \mathbb{N}^n} g(\mathbf{p}) \mathbf{x}^{\mathbf{p}}$ is a rational function,

we have the implications

$$A \Rightarrow B \Leftrightarrow C.$$

Thank You!