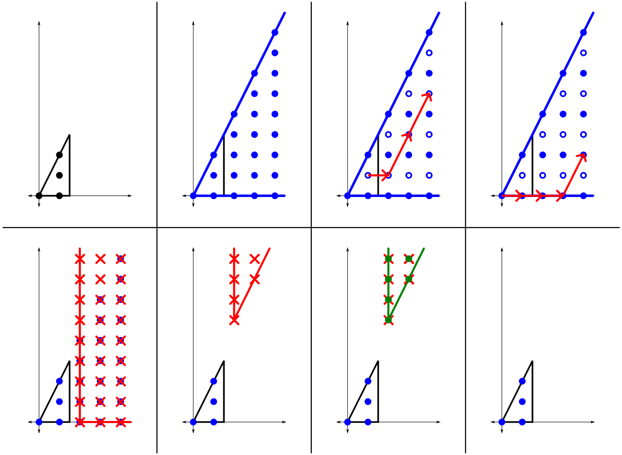
A Plethora of Polynomials: A Toolbox for Counting Problems using Presburger Arithmetic

Kevin Woods
Oberlin College



Quasi-polynomials

Definition: $g : \mathbb{N} \to \mathbb{N}$ is a quasi-polynomial of period m if there exist polynomials $g_0, g_1, \ldots, g_{m-1}$ such that

$$g(t) = g_{t \bmod m}(t), \forall t \in \mathbb{N}.$$

Example: For $t \in \mathbb{N}$, let

$$S_t = \{x \in \mathbb{N} : \ 1 \leq 2x \leq t\} = \{1, 2, \ldots, \lfloor t/2 \rfloor\}.$$

Then

$$|S_t| = \left\lfloor \frac{t}{2} \right\rfloor = \begin{cases} t/2, & \text{if } t \bmod 2 = 0, \\ (t-1)/2, & \text{if } t \bmod 2 = 1. \end{cases}$$

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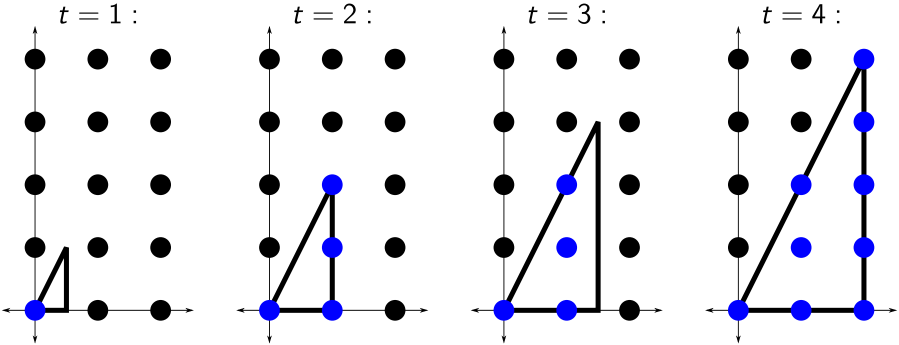
A Triangle

Let P be the triangle with vertices $(0,0)$, $(1/2,0)$, and $(1/2,1)$.

Let $S_t = tP \cap \mathbb{Z}^2$, for $t \in \mathbb{N}$.
What is $|S_t|$, as a function of t?



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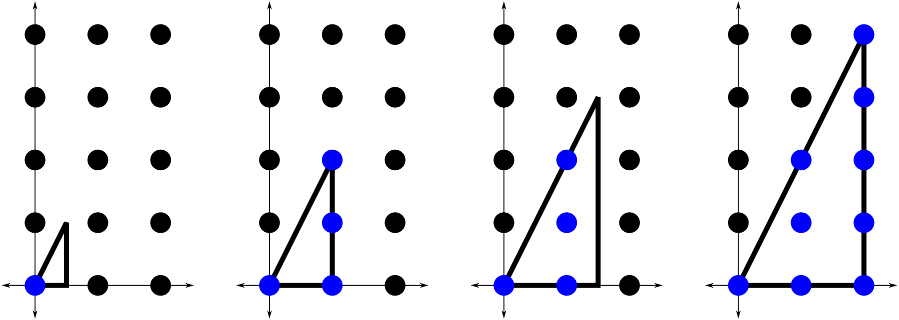

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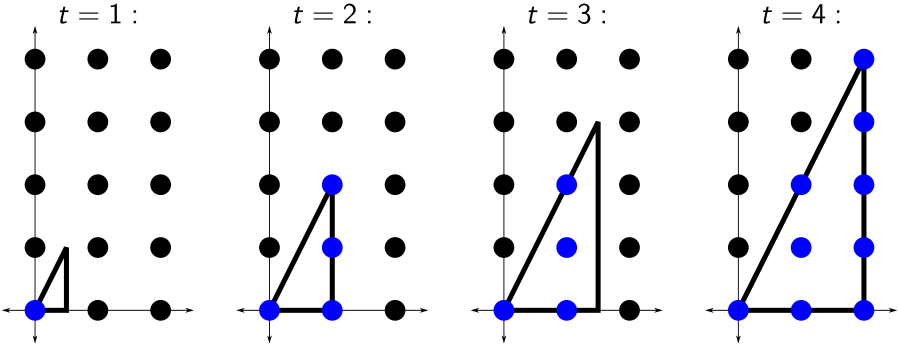

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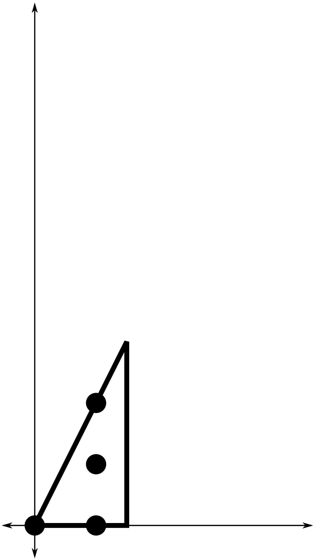

The hard (but insightful) way to calculate $|S_t|$:

Definition: The generating function for $S \subseteq \mathbb{Z}^2$ is given by

$$f(S; x, y) = \sum_{(c,d) \in S} x^c y^d.$$

Example:

$$f(S_3; x, y) = x^0y^0 + x^1y^0 + x^1y^1 + x^1y^2.$$

Let's find $f(S_t; x, y)$.

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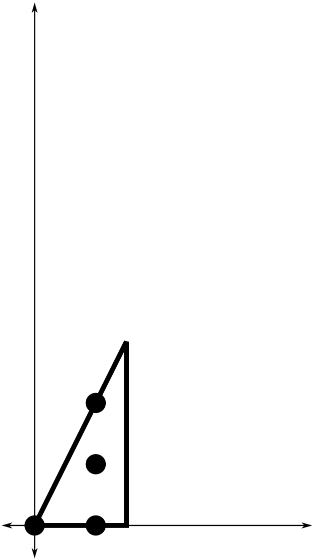

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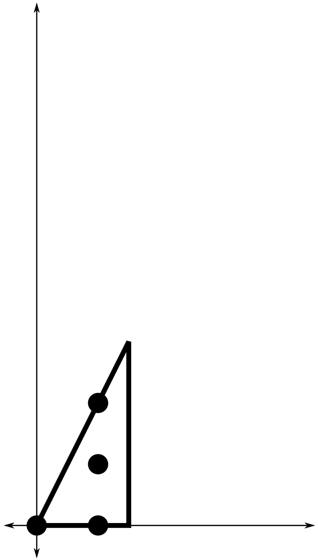

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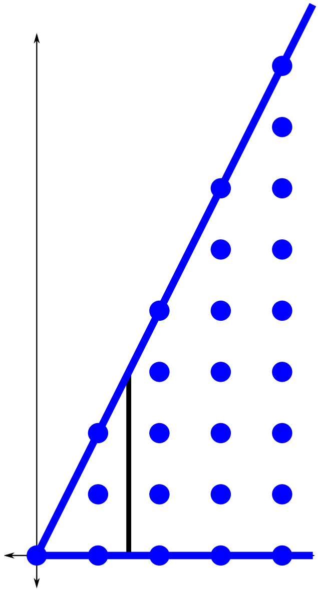

Let's first find $f(S; x, y)$ for this set.

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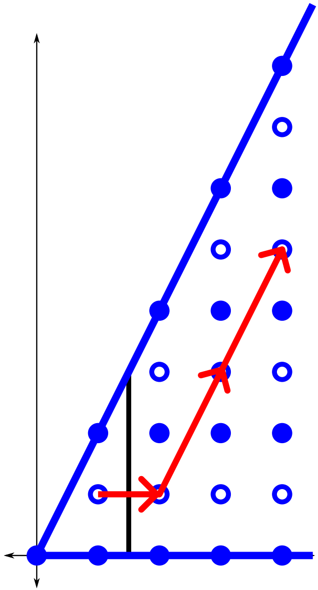

$f(S; x, y) =$

$(x^0 y^0 + x^1 y^1)$

$\cdot (1 + x^1 + x^2 + x^3 + \cdots)$

$\cdot (1 + (x^1 y^2)^1 + (x^1 y^2)^2 + \cdots)$

$x^4 y^5 = x^1 y^1 (x)^1 (x^1 y^2)^2$

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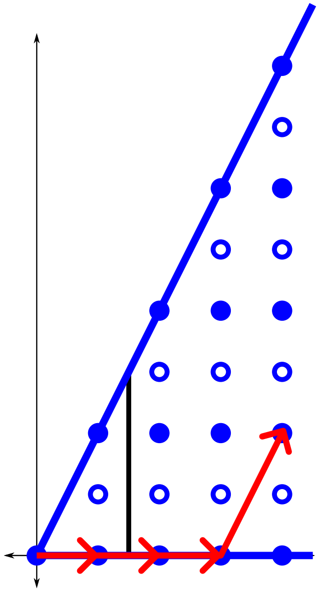

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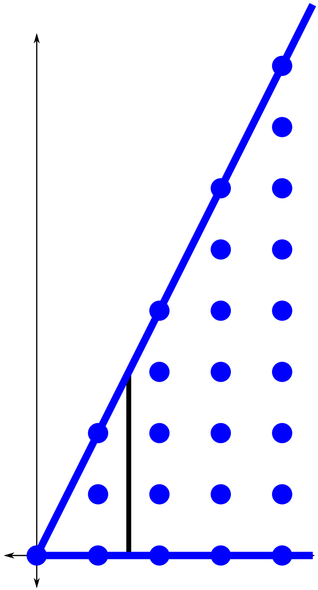

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$$\cdot (1 + x^1 + x^2 + x^3 + \cdots)$$
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$$= \frac{1 + xy}{(1 - x)(1 - xy^2)}$$

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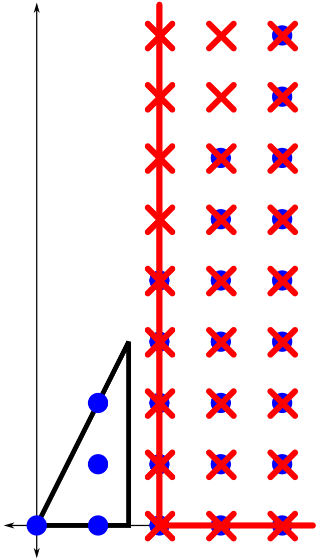

Let $k = \lfloor t/2 \rfloor$.

$$-x^{k+1}y^0 \cdot (1 + x + x^2 + \cdots) \cdot (1 + y + y^2 + \cdots)$$

$$= -\frac{x^{k+1}}{(1-x)(1-y)}$$

Only the vertex of the cone depends on t.

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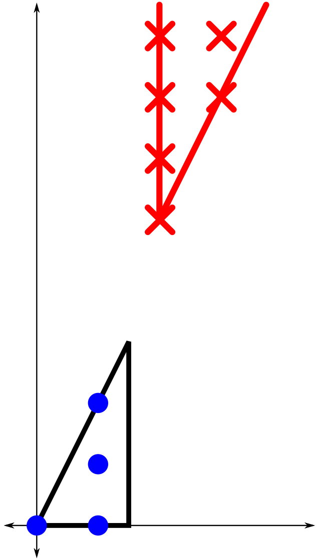

$$\frac{1+xy}{(1-x)(1-xy^2)}$$

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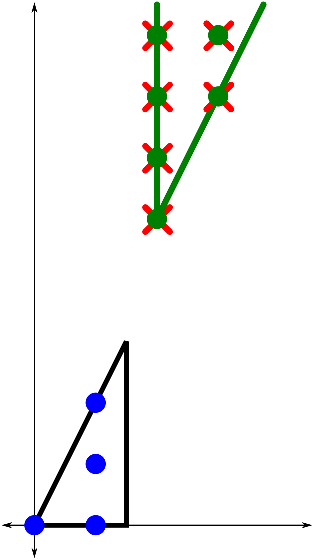

$$+x^{k+1}y^{2(k+1)+1}$$
$$\cdot\left(1+xy^2+(xy^2)^2+\cdots\right)$$
$$\cdot\left(1+y+y^2+\cdots\right)$$

$$=\frac{x^{k+1}y^{2k+3}}{(1-xy^2)(1-y)}$$

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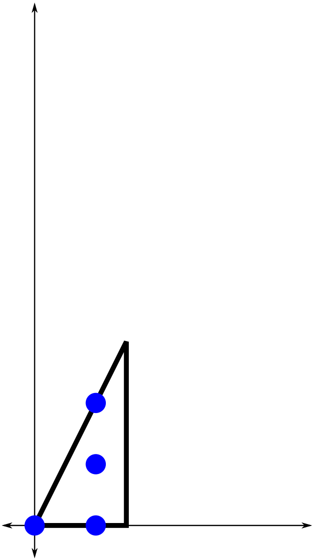

$$f(S_t; x, y) = \frac{1 + xy}{(1 - x)(1 - xy^2)} - \frac{x^{k+1}}{(1 - x)(1 - y)} + \frac{x^{k+1} y^{2k+3}}{(1 - xy^2)(1 - y)}.$$

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Parametric Polyhedra

Definition: A parametric polyhedron, $P_t \subseteq \mathbb{R}^d$, is the solution set to a system of linear inequalities of the form

$$a_1 x_1 + \cdots + a_d x_d \leq bt + c.$$

Theorem (Ehrhart, McMullen, Brion, Barvinok)

$|P_t \cap \mathbb{Z}^d|$ agrees with a quasi-polynomial, for sufficiently large t.

- Inclusion-exclusion on generating functions reduces to cones.
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Example: Let $R = k[x_1, x_2, x_3]$, graded so that $\deg x_i = a_i$.

$$\begin{aligned} f(t) :&= \dim_k\{p \in R : \ p \text{ homogeneous of degree } t\} \\ &= \left|\{x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \text{ of degree } t\}\right| \\ &= \left|\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 : \ \lambda_i \geq 0, \ a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = t\}\right| \end{aligned}$$

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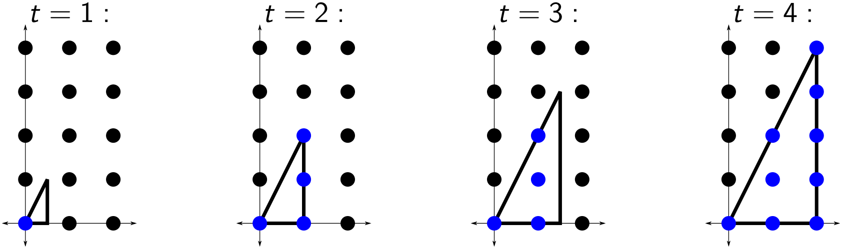
Boolean Operations

Our Example:

$$(x, y) \in \mathbb{Z}^2 : \ (y \geq 0) \land (2x \leq t) \land (y - 2x \leq 0)$$



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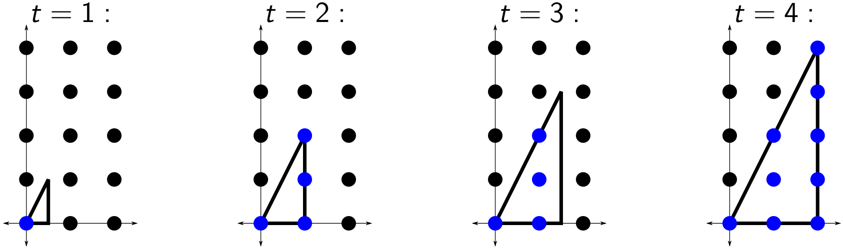

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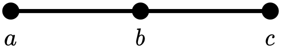
No problem [Barvinok–Pommersheim].
For example, Disjunctive Normal Form yields union of parametric polyhedra:

$$A \wedge (B \vee C \vee D) \quad \text{is} \quad (A \wedge B) \vee (A \wedge C) \vee (A \wedge D).$$

Boolean Operations

Let G be this graph:



Let $\chi_G(t)$ be the number of ways to color the vertices of G with t possible colors, so that no adjacent vertices have the same color. Then $\chi_G(t) = t(t-1)^2$, a polynomial.

If x_a, x_b, x_c are the colors of a, b, and c, then

- $1 \leq x_a, x_b, x_c \leq t$,
- $x_a \neq x_b$ and $x_b \neq x_c$,

a Boolean combination of linear (in)equalities.

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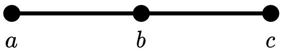

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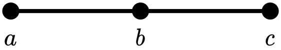

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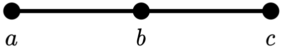

Let $\chi_G(t)$ be the number of ways to color the vertices of G with t possible colors, so that no adjacent vertices have the same color. Then $\chi_G(t) = t(t-1)^2$, a polynomial.

If x_a, x_b, x_c are the colors of a, b, and c, then

- $1 \leq x_a, x_b, x_c \leq t$,
- $x_a \neq x_b$ and $x_b \neq x_c$,

a Boolean combination of linear (in)equalities.

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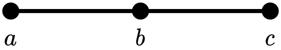

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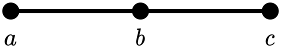

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Fill a 3×3 square with distinct positive integers such that the sum of every row, column, and two main diagonals are all exactly t.

12	1	11
7	8	9
5	15	4

If $t \equiv 6 \pmod{18}$, for example, then there are $\frac{2}{9}(t-6)(t-10)$ ways to do this [Beck–Zaslavsky]. A quasi-polynomial!

If x_{ij} is the number in the ith row and jth column,

- We require $x_{11} \neq x_{12}$ and so on,
- $x_{11} + x_{12} + x_{13} = t$ and so on,

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How many ways are there to place three queens on a $t \times t$ board such that no two queens are attacking each other? There are

$$\frac{t^6}{6} - \frac{5t^5}{3} + \frac{79t^4}{12} - \frac{25t^3}{2} + 11t^2 - \frac{43t}{12} + \frac{1}{8} + (-1)^t\left(\frac{t}{4} - \frac{1}{8}\right)$$

ways, a quasi-polynomial of period 2 [Chaiken–Hanusa–Zaslavsky]. Suppose we place Queen i at position $(x_i, y_i) \in \mathbb{Z}^2$ where $1 \leq x_i, y_i \leq t$. Then

- $x_1 \neq x_2$ says that the first two queens can't be in the same row,
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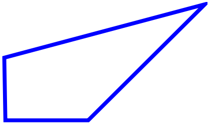

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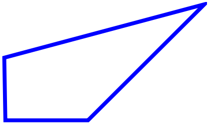

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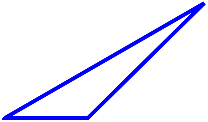

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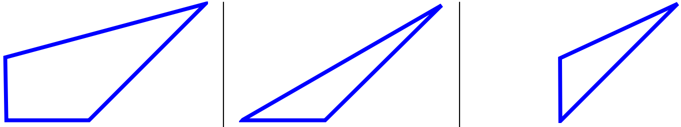

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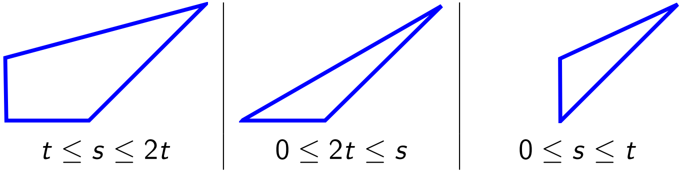

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As the combinatorial type changes, the inclusion-exclusion of cones changes. But for all polyhedra of the same combinatorial type, we're just in the same old setup.

End up with

$$c(s,t) = \begin{cases} \frac{s^2}{2} - \lfloor \frac{s}{2} \rfloor s + \frac{s}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1, & \text{if } t \le s \le 2t, \\ st - \lfloor \frac{s}{2} \rfloor s - \frac{t^2}{2} + \frac{t}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1, & \text{if } 0 \le 2t \le s, \\ \frac{t^2}{2} + \frac{3t}{2} + 1, & \text{if } 0 \le s \le t, \end{cases}$$

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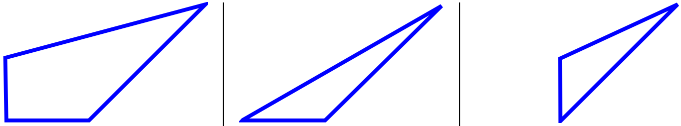

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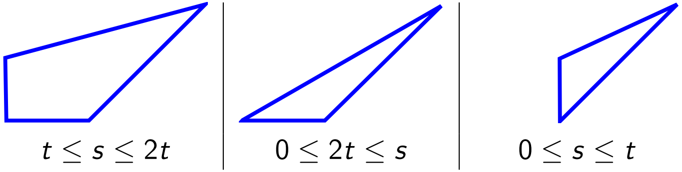

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$$\{x \in \mathbb{N} : \ \exists y \in \mathbb{N}, x = 3y + 1\} = \{x \in \mathbb{N} : \ x = 1 \bmod 3\}.$$

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Example of Quantifier Elimination: Let S be the set of degrees appearing in $k[x^3, x^5]$, that is, the semigroup generated by 3 and 5,

$$\begin{aligned} S &= \{0, 3, 5, 6, 8, 9, 10, \ldots\} \\ &= \{n \in \mathbb{Z} : \; \exists x, y \in \mathbb{Z}, \; (x \geq 0) \; \wedge \; (y \geq 0) \; \wedge \; (3x + 5y = n)\} \end{aligned}$$

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$$S = \{n \in \mathbb{Z} :\ \exists x, y \in \mathbb{Z},\ (x \geq 0)\ \wedge\ (y \geq 0)\ \wedge\ (3x + 5y = n)\}$$

If there exists any x satisfying this, then:

- It must be $x = (n - 5y)/3$,
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Presburger Arithmetic

Theorem (W)

Suppose F is a first-order formula over the integers, defined using linear inequalities, Boolean operations, and quantifiers, that is, F is a formula in Presburger arithmetic. Suppose the free (unquantified) variables in F are $c_1, \ldots c_d$ (the counted variables) and $p_1, \ldots, p_n$ (the parameter variables). Then

$$g(p_1, \ldots, p_n) = \#(c_1, \ldots, c_d) \text{ making } F \text{ true}$$

is a piecewise quasi-polynomial, defined on polyhedral pieces.

This covers a wide variety of counting problems in different fields.

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A Twist

Let's get greedy and go further. In what other settings do we still end up with quasi-polynomial behavior? Here's one:

- Require a single parameter, t.
- Allow multiplication by this parameter (but not by other variables).
- So base inequalities are of the form

$$p_1(t)x_1 + \cdots + p_n(t)x_n \leq q(t),$$

  where $p_i, q \in \mathbb{Z}[t]$. For fixed t, these are just linear inequalities.
- As t changes, normal vectors can "twist".
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Then you still get quasi-polynomials! (for sufficiently large t) [Bogart–Goodrick–W].

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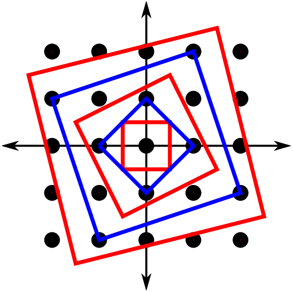
$$P_t = \Big\{(x,y) \in \mathbb{R}^2 : \; -(t^2 - 2t + 2) \le 2x + (2t - 2)y \le t^2 - 2t + 2, \quad -(t^2 - 2t + 2) \le (2 - 2t)x + 2y \le t^2 - 2t + 2\Big\};$$



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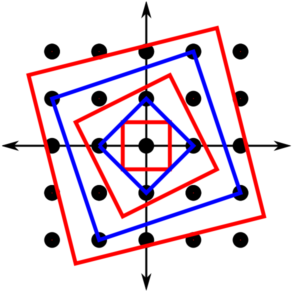

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Example: Let S_t be the degrees not appearing in $k\left[x^t, x^{t+1}, x^{t+3}\right]$.

$S_t = \{n \in \mathbb{Z} : \nexists a, b, c \in \mathbb{Z},\ a, b, c \geq 0,\ ta + (t+1)b + (t+3)c = n\}$.

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$$|S_t| = \begin{cases} \frac{1}{6}t^2 + \frac{1}{2}t, & \text{if } t \equiv 0 \pmod 3, \\ \frac{1}{6}t^2 + \frac{1}{2}t - \frac{2}{3}, & \text{if } t \equiv 1 \pmod 3, \\ \frac{1}{6}t^2 + \frac{1}{2}t - \frac{2}{3} & \text{if } t \equiv 2 \pmod 3. \end{cases}$$

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How Far is Too Far?

Let's get greedy and go further.

- Two nonlinear parameters is too far:

$$\begin{aligned} S_{s,t} &= \{(x,y) \in \mathbb{Z} : \; x, y \geq 0, \; sx + ty = st\} \\ &= \text{ the interval between } (0,s) \text{ and } (t,0). \\ |S_{s,t}| &= \gcd(s,t) + 1, \text{ not a quasi-polynomial.} \end{aligned}$$

- Nonlinearity in other variables is too far:

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Stay Greedy!

There is a cottage industry of finding examples that seem to have bad nonlinearity, and finding ways to preprocess so that they fit in this Presburger umbrella:

- Integer hull of a parametric polyhedron, $\mathrm{conv}(P_t \cap \mathbb{Z}^d)$ [Calegari–Walker],
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Open Question: How greedy can we get? Are there broader settings where quasi-polynomial behavior is guaranteed?

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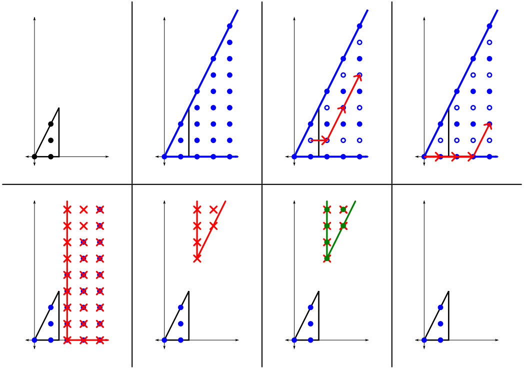
Thank You!



To see more details, check out:
Bogart–W, A plethora of polynomials: a toolbox for counting problems, The American Mathematical Monthly (2022), and references therein.