QUOTIENT MAPS WITH CONNECTED FIBERS AND THE FUNDAMENTAL GROUP

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Abstract. In classical covering space theory, a covering map induces an injection of fundamental groups. This note reveals a dual property for quotient maps having connected fibers, with applications to orbit spaces of smooth vector fields and leaf spaces in general.

1. Introduction

The opening question in Arnold's *Problems on singularities and dynamical systems* [1] asks whether an exotic $\mathbb{R}^4$ may appear as the orbit space of a polynomial or trigonometric vector field on $\mathbb{R}^5$. If one asks merely for a smooth vector field, then the answer is affirmative for every exotic $\mathbb{R}^4$ [1, p. 251]. Thus the gist of the question is: *does dynamics produce exotic differentiable manifolds in the simplest possible scenario?* This harks back to the quadratic polynomial vector field on $\mathbb{R}^3$ producing the Lorenz attractor: chaos is exhibited by a continuous dynamical system in the simplest possible setting.

Inspired by Arnold's question, it is natural to ask: for a given manifold M, which manifolds (smooth or topological) or spaces in general may appear as orbit spaces of vector fields (polynomial, smooth, or Lipschitz) on M? Of course, a non-closed orbit yields a non-Hausdorff orbit space. Still, nontrivial smooth manifolds arise: $\mathbb{C}P^{n-1}$ is already the orbit space of a quadratic polynomial vector field on $\mathbb{R}^{2n-1}$, as shown below in Section 3.6.

As a first step to addressing the above questions, we prove the following basic result.

Lemma. *Let X be a locally path connected topological space partitioned into connected subsets (equivalence classes). Let $\pi : X \to X/\!\sim$ be the associated quotient map. If $X/\!\sim$ is semi-locally simply connected, then the induced homomorphism*

$$\pi_\sharp : \pi_1(X) \to \pi_1(X/\!\sim)$$

of fundamental groups is surjective for each choice of basepoint in X.

No separation properties are assumed in this lemma. Intuitively, the lemma says that π_1 may not be created by collapsing connected subsets to points. The lemma is optimal in general as shown by explicit examples in Section 3. Each example satisfies two of the three hypotheses in the lemma, with the conclusion of the lemma

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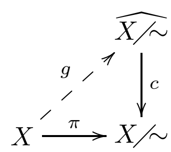
failing. We take a moment now to outline these examples.

Section 3.1 presents probably the simplest example with a single element of the partition not connected. Section 3.2 uses the well-known Warsaw circle to construct an example that is not locally path connected. In this example each element of the partition is connected as required, although one element is not path connected. Section 3.3 modifies the previous example to obtain one where again X is not locally path connected but now every element of the partition is path connected. Section 3.4 constructs a pair of spaces (HR, A) with the following properties. The space HR, which we call the Hawaiian ropes, is Hausdorff, path connected, locally contractible, paracompact (hence normal), homotopy equivalent to the wedge of countably many circles, and not metrizable. The subspace A is closed in HR, has all homotopy groups trivial (i.e. is weakly contractible) but is not contractible. The quotient HR/A is homeomorphic to the well-known Hawaiian earring HE which is not semi-locally simply connected. Thus $\pi_\sharp : \pi_1(\mathrm{HR}) \to \pi_1(\mathrm{HR}/A)$ maps from a countable group to an uncountable one, failing radically to be surjective. This example raises the question of whether X must have a nontrivial fundamental group in order to create new π_1 in the quotient; in fact it need not. Section 3.5 uses the pair (HR, A) to construct a pair (X, A') with the following properties. The space X is contractible, locally contractible, Hausdorff, normal, and not metrizable. The subspace A' is closed in X and is weakly contractible. The quotient X/A' has an uncountable first integral homology group, in particular $\pi_1(X/A')$ is (highly!) nontrivial. We present these examples in detail as they reveal the optimality of the lemma.

The proof of the lemma makes essential use of a lift of π to an appropriate covering space of $X/{\sim}$.



Rigid covering fibrations [2] generalize classical covering spaces, the key hypothesis being that the topological fundamental group is totally disconnected rather than discrete as in the classical theory (see [4]). Our example in Section 3.4 shows that rigid covering fibrations may not be used to weaken the semi-local simple connectivity hypothesis in the lemma. This may be further explained as follows. A rigid covering fibration does not in general have local inverses: consider a simply connected rigid covering fibration of HE and an arbitrary open set in HE containing the wild point. Local inverses are used in classical covering space theory to construct lifts such as g above.

For a familiar special case of the lemma, consider the projection $p : E \to B$ of a fibration. The associated long exact sequence shows that if the fibers are path connected, then $p_\sharp : \pi_1(E) \to \pi_1(B)$ is surjective. The above lemma is more general in that the fibers need not be homotopy equivalent. On the other hand, the lemma is special to the fundamental group in that it does not directly extend to higher homotopy groups. Section 3.7 constructs a partition of closed upper 3–space (a contractible space) into connected arcs (one open, the rest closed, and all contractible) where the quotient space is homeomorphic to the 2–sphere. The results in Section 3.6 explicitly exhibit this quotient as the orbit space of a quadratic vector field restricted to a linear half-space. Thus it is analytically very simple.

Let M be a smooth n–dimensional manifold that is connected, Hausdorff, and separable. Let $v : M \to TM$ be a smooth vector field on M. Integrating v yields a partition of M into connected orbits, each of which is an injective image of a point, an open interval, or the unit circle. In this case $M/\!\sim$ is called the *orbit space* of v and the natural quotient map is $\pi : M \to M/\!\sim$. The lemma implies that if $M/\!\sim$ is semi-locally simply connected, then $\pi_\sharp$ is surjective. In particular, the lemma restricts which manifolds may arise as $M/\!\sim$ for a given M. Manifolds arising as orbit spaces of $\mathbb{R}^n$, for example, must be simply connected. Similar results apply to p–dimensional foliations of M and associated leaf spaces.

The question arises whether every such orbit space $M/\!\sim$ is semi-locally simply connected. The authors know of no counterexample.

2. Proof of the Lemma

Consider a topological space X and a partition $\mathcal{P} = \{X_i \mid i \in I\}$ of X where I is some index set. The associated equivalence relation on X is: $x \sim y$ if and only if $x, y \in X_i$ for some $i \in I$. Let $\overline{x}$ denote the equivalence class represented by x, $X/\!\sim$ the set of equivalence classes, and $\pi : X \to X/\!\sim$ the canonical surjection. The set $X/\!\sim$ is equipped with the quotient topology making π continuous.

For the proof of the lemma, we assume that X is locally path connected, X_i is connected for each $i \in I$, and $X/\!\sim$ is semi-locally simply connected.

As X is locally path connected, its components and path components coincide and are both open and closed in X. Each X_i is connected and π is surjective, so each component C of X is saturated and $\pi(C)$ is both open and closed in $X/\!\sim$. Thus choosing a basepoint in X amounts to simply restricting π to a path component of X. Therefore we can and do assume X itself is path connected. As π is a continuous surjection, $X/\!\sim$ is path connected as well.

Next, we show that $X/\!\sim$ is locally path connected. Note that $\pi^{-1}(E) \subset X$ is saturated for every $E \subset X/\!\sim$. Let U be an open neighborhood of $\overline{x}$ in $X/\!\sim$. Then $\pi^{-1}(U)$ is open in the locally path connected space X and so $\pi^{-1}(U)$ is locally path connected. Thus the components of $\pi^{-1}(U)$ coincide with its path components, are open in $\pi^{-1}(U)$, and hence are open in X. As $\pi^{-1}(U)$ is saturated and fibers of π are connected, we see that each component of $\pi^{-1}(U)$ is saturated. Let V be the component of $\pi^{-1}(U)$ containing the fiber $\pi^{-1}(\overline{x})$. Then $\pi^{-1}(\pi(V)) = V$ and so $\pi(V) \subset U$ is an open path connected neighborhood of $\overline{x}$ as desired.

By hypothesis, $X/\!\sim$ is semi-locally simply connected and so classical covering space theory applies to $X/\!\sim$. To the subgroup $\pi_\sharp\left(\pi_1(X)\right)$ of $\pi_1\left(X/\!\sim\right)$ there corresponds the commutative diagram

(1)
$$\begin{array}{ccc} & & \widehat{X/\!\sim} \\ & \overset{g}{\nearrow} & \downarrow c \\ X & \xrightarrow{\pi} & X/\!\sim \end{array}$$

where c is a covering map, the induced homomorphisms of fundamental groups satisfy

(2)
$$\mathrm{Im}(c_\sharp) = \mathrm{Im}(\pi_\sharp),$$

and g is a lift of π. We will show that the restriction $c|_{\mathrm{Im}(g)}$ of c to the image of g is a homeomorphism onto $X/\!\sim$.

Surjectivity of $c|_{\mathrm{Im}(g)}$ is trivial since π is surjective. For injectivity, note that fibers of c are discrete and fibers of π are connected. Commutativity of (1) implies that g maps each fiber of π to a fiber of c. Hence g is constant on each fiber of π and $c|_{\mathrm{Im}(g)}$ is injective.

Next we show that $\mathrm{Im}(g)$ is open in $\widehat{X/\!\sim}$. Let $g(x) \in \mathrm{Im}(g)$. Then $\overline{x} \in X/\!\sim$ lies in a connected open set U evenly covered by c. Note that $c^{-1}(U)$ is the disjoint union
$$c^{-1}(U) = \coprod_{j \in J} U_j$$
of connected open sets in $\widehat{X/\!\sim}$ where J is some index set, $c|_{U_j} : U_j \to U$ is a homeomorphism for each $j \in J$, and $g(x) \in U_0$. The connected components of $\pi^{-1}(U)$ are open and saturated. So, their images under π are disjoint and open. Therefore $\pi^{-1}(U)$ is connected, saturated, and open in X. It follows that g maps $\pi^{-1}(U)$ into U_0. To see g maps $\pi^{-1}(U)$ onto U_0, let $y \in U_0$. Then $c(y) \in U$ and so there is $z \in \pi^{-1}(U)$ such that $\pi(z) = c(y)$. Now $g(z) \in U_0$ and commutativity of (1) yields $c(g(z)) = \pi(z) = c(y)$. But $c|_{U_0} : U_0 \to U$ is a homeomorphism and so $g(z) = y$ as desired. Hence $g(\pi^{-1}(U)) = U_0$, which is an open neighborhood of $g(x)$ in $\widehat{X/\!\sim}$. Whence $\mathrm{Im}(g)$ is open in $\widehat{X/\!\sim}$.

As c is a local homeomorphism and $\mathrm{Im}(g)$ is open, we have $c|_{\mathrm{Im}(g)}$ is a local homeomorphism. As $c|_{\mathrm{Im}(g)}$ is bijective, it is a homeomorphism onto $X/\!\sim$.

Finally, let $\alpha : [0,1] \to X/\!\sim$ be a based loop. Then $\hat{\alpha} = c|^{-1}_{\mathrm{Im}(g)} \circ \alpha$ is a based loop in $\widehat{X/\!\sim}$ and $c_\sharp\left([\hat{\alpha}]\right) = [\alpha]$. Therefore $c_\sharp$ is surjective and, by (2), $\pi_\sharp$ is surjective. The proof of the lemma is complete.

Remark 1. *As c is a covering map, $c_\sharp$ is injective. Thus the lemma implies that $c_\sharp$ is an isomorphism and c is a one sheeted cover. By a covering space isomorphism we may take $\widehat{X/\!\sim} = X/\!\sim$ and $c = \mathrm{id}$. By commutativity of (1), $\pi = g \circ \mathrm{id} = g$. It follows that g is surjective. The proof of the lemma may be reorganized to argue g is surjective directly.*

3. Examples

This section contains the examples outlined in the introduction. Sections 3.1–3.5 demonstrate the necessity of each hypothesis in the lemma. The titles of those sections indicate the relevant hypothesis.

3.1. **Connectedness of partition elements.** The simplest example is $X = [0,1] \subset \mathbb{R}$ and $\mathcal{P} = \{\{0,1\}\} \cup \{\{p\} \mid p \in (0,1)\}$ where just one element of $\mathcal{P}$ is not connected. The quotient $X/\!\sim$ is homeomorphic to S^1 and so $\pi_\sharp$ is not surjective. For more examples, consider any nontrivial classical covering map.

3.2. **Local path connectedness I.** Consider the Warsaw circle W shown in Figure 1 which is the subspace of $\mathbb{R}^2$ consisting of the points in the sets

$$A = \{(0,y) \mid -1 \le y \le 1\},$$
$$B = \{(x, \sin(1/x)) \mid 0 < x \le 1/\pi\}, \text{ and}$$
$$C = \{(x,0) \mid -2 \le x < 0 \text{ or } 1/\pi < x \le 2\} \cup \big\{(x,y) \mid x^2 + y^2 = 4 \ \&\ y \le 0\big\}.$$

The set B is a portion of the topologists sine curve. The Warsaw circle is the

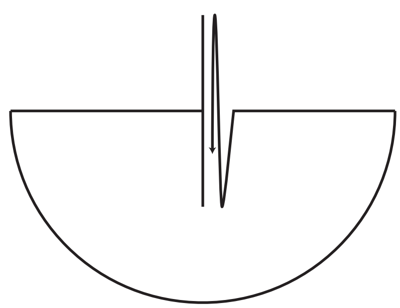


Figure 1. Warsaw circle $W \subset \mathbb{R}^2$.

classical example of a path connected but not locally path connected space. Notice that W is simply connected. Consider the partition $\mathcal{P} = \{A \cup B\} \cup \{\{c\} \mid c \in C\}$ of W into connected subsets. The quotient $W/\!\sim$ is homeomorphic to S^1 and so $\pi_\sharp : \pi_1(W) \to \pi_1(W/\!\sim)$ is not surjective.

3.3. **Local path connectedness II.** This example serves the same purpose as the previous one but it enjoys the additional property that every element of the partition is path connected and not just connected. Consider the Warsaw circle W and the partition $\mathcal{P} = \{A, B\} \cup \{\{c\} \mid c \in C\}$ of W into path connected subsets. Let $\pi : W \to W/\!\sim$ be the associated quotient map. While $W/\!\sim$ and S^1 are not homeomorphic (the former is not Hausdorff), we will show that they are homotopy equivalent thus completing the example.

Define the subspace $V \subset W$ shown in Figure 2 as follows

$$V = A \cup B \cup \{(x,0) \mid -2 \le x < 0 \text{ or } 1/\pi < x \le 2\}.$$

Note that V is closed and saturated in W. It follows that $\pi(V)$ is a closed subspace of $W/\!\sim$ and the restriction $\pi| : V \to \pi(V)$ is a quotient map.

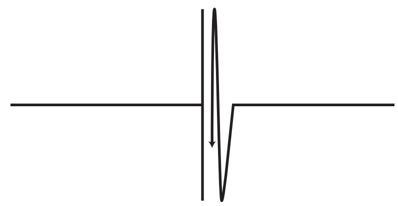


FIGURE 2. Closed subspace $V \subset W$.

Let $J = [-2, 2]$ and consider the equivalence relation $\sim$ on J generated by $x \sim 1/\pi$ for $0 < x \leq 1/\pi$. The quotient map is $\alpha : J \to J/\!\sim$. Define

$$\begin{array}{rcl} V & \xrightarrow{\tilde{f}} & J \\ (x, y) & \longmapsto & x \end{array}$$

which is a continuous surjection. As V is compact and J is Hausdorff, $\tilde{f}$ is a closed map. So $\tilde{f}$ is a quotient map and the composition $\alpha \circ \tilde{f}$ is a quotient map as well. The function $\alpha \circ \tilde{f}$ factors as follows

$$\begin{array}{ccc} V & \xrightarrow{\tilde{f}} & J \\ {\scriptstyle \pi|}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \pi(V) & \xrightarrow{f} & J/\!\sim \end{array} \tag{3}$$

to yield a (unique) continuous map $f : \pi(V) \to J/\!\sim$ making (3) commute. More explicitly, one verifies that $\alpha \circ \tilde{f}$ is constant on each fiber $\pi|^{-1}(\overline{p})$ where $\overline{p} \in \pi(V)$; only points with nontrivial fibers need to be checked, here they are $\overline{p} = \overline{(0, 0)}$ and $\overline{p} = \overline{(1/\pi, 0)}$ and the verification is easy. Then the universal property of quotient maps [6, Thm. 22.2] implies that the unique set function f making (3) commute is in fact continuous. This universal property will often be used below. Presently f is a bijection and so the universal property implies f is a homeomorphism.

Let $K = [-2, 2]$ and consider the equivalence relation $\sim$ on K generated by $x \sim 1$ for $0 < x \leq 1$. The quotient map is $\beta : K \to K/\!\sim$. The obvious piecewise linear homeomorphism

$$\begin{array}{rcll} K & \xrightarrow{\tilde{s}} & J & \\ x & \longmapsto & x & -2 \leq x \leq 0 \\ x & \longmapsto & \frac{1}{\pi}x & 0 < x \leq 1 \\ x & \longmapsto & \left(2 - \frac{1}{\pi}\right)x - 2 + \frac{2}{\pi} & 1 < x \leq 2 \end{array}$$

yields the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\tilde{s}} & J \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ K/\!\sim & \xrightarrow{s} & J/\!\sim \end{array}$$

where s is a homeomorphism. Using K instead of J simply eases notation in the proof of the following key claim.

Claim 1. *The spaces K and $K/{\sim}$ are homotopy equivalent.*

Proof. We already have the quotient map

$$\begin{array}{rcll} K & \xrightarrow{\beta} & K/{\sim} & \\ x & \longmapsto & \overline{x} & -2 \leq x \leq 0 \text{ or } 1 < x \leq 2 \\ x & \longmapsto & \overline{1} & 0 < x \leq 1 \end{array}$$

Define

$$\begin{array}{rcll} K & \xrightarrow{\tilde{g}} & K & \\ x & \longmapsto & x & -2 \leq x \leq 0 \\ x & \longmapsto & 0 & 0 < x \leq 1 \\ x & \longmapsto & 2x-2 & 1 < x \leq 2 \end{array}$$

which is a continuous surjection. The map $\tilde{g}$ factors as follows

$$\begin{array}{ccc} K & \xrightarrow{\tilde{g}} & K \\ {\scriptstyle\beta}\downarrow & \nearrow_{g} & \\ K/{\sim} & & \end{array}$$

to yield the continuous surjection

$$\begin{array}{rcll} K/{\sim} & \xrightarrow{g} & K & \\ \overline{x} & \longmapsto & x & -2 \leq x \leq 0 \\ \overline{1} & \longmapsto & 0 & \\ \overline{x} & \longmapsto & 2x-2 & 1 < x \leq 2 \end{array}$$

We will show that β and g are homotopy inverses by constructing homotopies

$$\widetilde{H} : K \times I \to K$$
$$H : K/{\sim} \times I \to K/{\sim}$$

such that

$$\widetilde{H}_0 = g \circ \beta \text{ and } \widetilde{H}_1 = \mathrm{id}_K \tag{4}$$
$$H_0 = \beta \circ g \text{ and } H_1 = \mathrm{id}_{K/{\sim}} \tag{5}$$

Define

$$\begin{array}{rcll} K \times I & \xrightarrow{\widetilde{H}} & K & \\ (x,t) & \longmapsto & x & -2 \leq x \leq 0 \\ (x,t) & \longmapsto & t \cdot x & 0 < x \leq 1 \\ (x,t) & \longmapsto & (2-t)x + 2t - 2 & 1 < x \leq 2 \end{array}$$

which is continuous. As $\widetilde{H}_0 = \tilde{g} = g \circ \beta$, (4) is satisfied.

Consider the diagram

$$(6)\qquad \begin{array}{ccc} K\times I & \xrightarrow{\widetilde{H}} & K \\ {\scriptstyle \beta\times\mathrm{id}_I}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ K/{\sim}\times I & \overset{H}{\dashrightarrow} & K/{\sim} \end{array}$$

As β is a quotient map and I is (locally) compact and Hausdorff, $\beta\times\mathrm{id}_I$ is a quotient map. The fiber $(\beta\times\mathrm{id}_I)^{-1}(\overline{x},t)=\beta^{-1}(\overline{x})\times\{t\}$. So, the universal property applies provided $\beta\circ\widetilde{H}$ is constant on $\beta^{-1}\left(\overline{1}\right)\times\{t\}=(0,1]\times\{t\}$ for each $t\in I$. If $t=0$, then $\widetilde{H}_0((0,1])=\{0\}$. If $0<t\leq 1$, then $\widetilde{H}_t((0,1])\subset(0,1]$ and β is constant on $(0,1]$. Thus the unique set function H making (6) commute is continuous. By (6) and (4) we have

$$\begin{aligned} H_0\circ\beta &= \beta\circ\widetilde{H}_0=\beta\circ(g\circ\beta)=(\beta\circ g)\circ\beta \text{ and}\\ H_1\circ\beta &= \beta\circ\widetilde{H}_1=\beta\circ\mathrm{id}_K=\mathrm{id}_{K/\sim}\circ\beta \end{aligned}$$

and, since β is surjective, (5) holds. For future reference, note that for every $t\in I$ we have

$$(7)\qquad \widetilde{H}(-2,t)=-2 \text{ and } \widetilde{H}(2,t)=2$$

and so by (6)

$$(8)\qquad H\left(\overline{-2},t\right)=\overline{-2} \text{ and } H\left(\overline{2},t\right)=\overline{2}.$$

□

Claim 2. *The spaces $W/{\sim}$ and S^1 are homotopy equivalent.*

Proof. Let $S=\left\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=4\right\}$ and let S^+ and S^- denote the upper and lower closed semicircles in S respectively. Thus $S^+\cap S^-=\{(\pm 2,0)\}$. We will show that $W/{\sim}$ and S are homotopy equivalent.

Let $\sigma: K\to S^+$ be any homeomorphism with $\sigma(-2)=(-2,0)$ (consequently $\sigma(2)=(2,0)$). We have the following diagram where $\approx$ indicates a homeomorphism

$$\pi(V)\ \underset{\approx}{\xrightarrow{f}}\ J/{\sim}\ \underset{\approx}{\xleftarrow{s}}\ K/{\sim}\ \underset{\beta}{\overset{g}{\rightleftarrows}}\ K\ \underset{\approx}{\xrightarrow{\sigma}}\ S^+$$

Define

$$\begin{aligned} \varphi^+ &= \sigma\circ g\circ s^{-1}\circ f \quad\text{and}\\ \psi^+ &= f^{-1}\circ s\circ\beta\circ\sigma^{-1}. \end{aligned}$$

Let Φ^+ denote the composition of the following maps

$$\pi(V)\times I\xrightarrow{f\times\mathrm{id}_I}J/{\sim}\times I\xrightarrow{s^{-1}\times\mathrm{id}_I}K/{\sim}\times I\xrightarrow{H}K/{\sim}\xrightarrow{s}J/{\sim}\xrightarrow{f^{-1}}\pi(V).$$

Note that $\Phi^+_0=\psi^+\circ\varphi^+$, $\Phi^+_1=\mathrm{id}_{\pi(V)}$, and, using (8), Φ^+_t fixes $\overline{(-2,0)}$ and $\overline{(2,0)}$ for each $t\in I$. Let Ψ^+ denote the composition of the following maps

$$S^+\times I\xrightarrow{\sigma^{-1}\times\mathrm{id}_I}K\times I\xrightarrow{\widetilde{H}}K\xrightarrow{\sigma}S^+.$$

Note that $\Psi_0^+ = \varphi^+ \circ \psi^+$, $\Psi_1^+ = \mathrm{id}_{S^+}$, and, using (7), Ψ_t^+ fixes $(-2,0)$ and $(2,0)$ for each $t \in I$.

Now S^- is a closed and saturated subspace of $W \subset \mathbb{R}^2$ and $W/\!\sim$ is the union of the two closed subspaces $\pi(V)$ and $\pi(S^-)$. Their intersection

$$\pi(V) \cap \pi(S^-) = \left\{\overline{(\pm 2, 0)}\right\}$$

is closed in $W/\!\sim$. Define

$$\begin{array}{ccc} \pi(S^-) \xrightarrow{\varphi^-} S^- & \text{and} & S^- \xrightarrow{\psi^-} \pi(S^-) \\ \overline{p} \longmapsto p & & p \longmapsto \overline{p} \end{array}$$

which are homeomorphisms. Paste together φ^+ and φ^- to obtain $\varphi : W/\!\sim \to S$. Paste together ψ^+ and ψ^- to obtain $\psi : S \to W/\!\sim$. Both φ and ψ are continuous. Define

$$\begin{array}{ccc} \pi(S^-) \times I \xrightarrow{\Phi^-} S^- & \text{and} & S^- \times I \xrightarrow{\Psi^-} S^- \\ (\overline{p}, t) \longmapsto \overline{p} & & (p, t) \longmapsto p \end{array}$$

Paste together Φ^+ and Φ^- to obtain $\Phi : W/\!\sim \times I \to W/\!\sim$. Paste together Ψ^+ and Ψ^- to obtain $\Psi : S \times I \to S$. Both Φ and Ψ are continuous. An easy verification shows that

$$\Phi_0 = \psi \circ \varphi \text{ and } \Phi_1 = \mathrm{id}_{W/\sim}$$
$$\Psi_0 = \varphi \circ \psi \text{ and } \Psi_1 = \mathrm{id}_S$$

Thus φ and ψ are homotopy inverses proving the claim. □

3.4. **Semi-local simple connectedness.** Let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $I = [0,1] \subset \mathbb{R}$, and $I_n = I \times \{n\}$. Consider the disjoint union

$$\widetilde{X} = \coprod_{n \in \mathbb{N}_0} I_n = I \times \mathbb{N}_0 \tag{9}$$

and the equivalence relation $\sim$ on $\widetilde{X}$ generated by $(0,n) \sim (0,0)$ and $(1,n) \sim (1/n, 0)$ for $n \geq 1$. Define HR, which we call the Hawaiian ropes, to be the quotient space $X = \widetilde{X}/\!\sim$ as depicted in Figure 3. HR is noncompact and is not a subspace

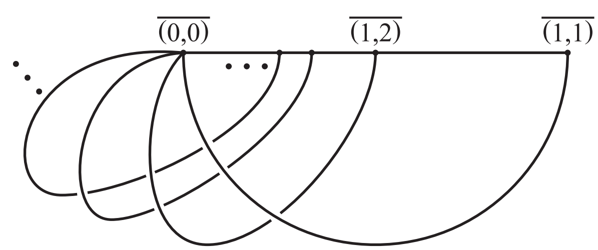


FIGURE 3. Hawaiian ropes HR $= X$.

of $\mathbb{R}^3$. Intuitively, the attached arcs are large and their interiors do not accumulate. Not to be confused with HR is the compact metric subspace of $\mathbb{R}^2$ shown in Figure 4.

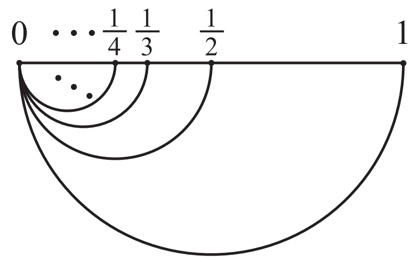


FIGURE 4. Compact metric subspace of $\mathbb{R}^2$.

Define

$$\widetilde{Y} = \widetilde{X} - ((0,1] \times \{0\})$$

and let $\sim$ be the equivalence relation on $\widetilde{Y}$ generated by $(0,n) \sim (0,0) \sim (1,n)$ for $n \geq 1$. The quotient space $Y = \widetilde{Y}/\!\sim$ is the wedge of countably many circles. Not to be confused with Y is the well known compact subspace $\mathrm{HE} \subset \mathbb{R}^2$ shown in Figure 5 commonly called the Hawaiian earring.

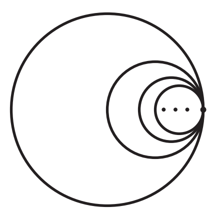


FIGURE 5. Hawaiian earring $\mathrm{HE} \subset \mathbb{R}^2$.

We have the quotient maps

$$\begin{aligned} \widetilde{X} &\xrightarrow{\alpha} X = \mathrm{HR} \\ x &\longmapsto \overline{x} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \widetilde{Y} &\xrightarrow{\beta} \mathrm{Y} \\ y &\longmapsto \overline{y} \end{aligned}$$

Define the closed subspace $A \subset X = \mathrm{HR}$, depicted in Figure 6, by

$$A = \bigcup_{n \geq 1} \alpha\left(I_n\right). \tag{11}$$

In other words, A consists of all of the curved arcs (ropes) depicted in Figure 3. Note that $\overline{(0,0)} = \overline{(0,1)}$ in X, but we will write $\overline{(0,1)}$ for the wild point in A since (11) alone does not make obvious that $\overline{(0,0)}$ lies in A.

We claim that $X = \mathrm{HR}$ is homotopy equivalent to Y and HR/A is homeomorphic to HE. Assume these claims for the moment. Then $\pi_1\left(\mathrm{HR}\right)$ is free on countably many generators, HR/A is not semi-locally simply connected, and $\pi_1\left(\mathrm{HR}/A\right)$ is uncountable (and not free) [7]. Thus the quotient map $\pi : \mathrm{HR} \to \mathrm{HR}/A$ does not induce a surjection of fundamental groups (far from it!). Below we prove these two

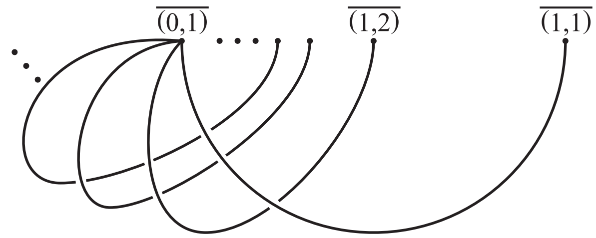


FIGURE 6. Subspace $A \subset X = \text{HR}$.

claims and others made in the introduction concerning $X = \text{HR}$ and A.

We begin by constructing a neighborhood basis of the wild point $\overline{(0,0)}$ in $X = \text{HR}$. If $N \in \mathbb{N}$ and $a = \{a_n\}$ and $b = \{b_n\}$ are sequences of real numbers such that

$$0 < a_n < \frac{1}{2} \text{ for each } n \geq 1 \text{ and} \tag{12}$$

$$\frac{1}{2} < b_n < 1 \text{ for each } n > N, \tag{13}$$

then we define the set

$$\widetilde{V}(N, a, b) \subset \widetilde{X} = I \times \mathbb{N}_0$$

as follows

$$\widetilde{V}(N, a, b) = ([0, 1/N) \times \{0\}) \cup \left(\bigcup_{n \geq 1} [0, a_n) \times \{n\} \right) \cup \left(\bigcup_{n > N} (b_n, 1] \times \{n\} \right).$$

Such a set is shown in Figure 7 with $N = 3$. Notice that each such $\widetilde{V}(N, a, b)$ is a

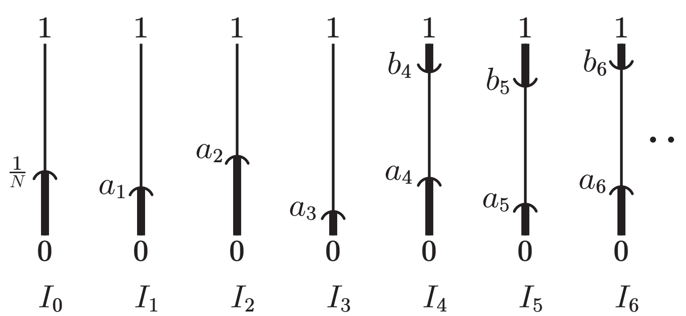


FIGURE 7. Saturated open set $\widetilde{V}(N, a, b) \subset \widetilde{X}$ with $N = 3$.

saturated open set in $\widetilde{X}$ containing $\alpha^{-1}\left(\overline{(0,0)}\right)$ and so

$$V(N, a, b) = \alpha\left(\widetilde{V}(N, a, b)\right) \tag{14}$$

is an open neighborhood of $\overline{(0,0)}$ in $X = \text{HR}$.

Claim 3. *The open sets $V(N, a, b)$ with $N \in \mathbb{N}$ and sequences a and b satisfying* (12) *and* (13) *form a neighborhood basis of $\overline{(0,0)}$ in $X = \text{HR}$.*

Proof. Let U be an open set in X containing $\overline{(0,0)}$. Then $\widetilde{U} = \alpha^{-1}(U)$ is a saturated open set in $\widetilde{X}$ containing $(0,n)$ for $n \in \mathbb{N}_0$. Thus there is $N \in \mathbb{N}$ such that $[0,1/N) \times \{0\} \subset \widetilde{U}$. Also there is a sequence a satisfying (12) such that $[0,a_n) \times \{n\} \subset \widetilde{U}$ for each $n \geq 1$. As $\widetilde{U}$ is saturated and contains $[0,1/N) \times \{0\}$, $\widetilde{U}$ contains $(1,n)$ for $n > N$. As $\widetilde{U}$ is open, there is a sequence b satisfying (13) such that $(b_n, 1] \times \{n\} \subset \widetilde{U}$. Therefore $V(N,a,b) \subset U$ as desired. □

Corollary 1. *The space* $X = \mathrm{HR}$ *is Hausdorff.*

Proof. This easy verification is left to the reader. □

Corollary 2. *The space* $X = \mathrm{HR}$ *is not first countable. In particular* $X = \mathrm{HR}$ *is not metrizable.*

Proof. Suppose $\{U_i\}_{i \in \mathbb{N}}$ is a countable neighborhood basis of $\overline{(0,0)}$ in X. By Claim 3 each U_i contains some basic open set

$$V_i = V\left(N(i), a(i), b(i)\right).$$

Using a diagonal argument we construct a basic open set $V(N,a,b)$ not containing any V_i. Let $N = 1$, let b be the constant sequence $b_n = 3/4$ for $n > 1$, and let a be the sequence defined by

$$a_n = \frac{1}{2} a(n)_n$$

for each $n \geq 1$. Then $V(N,a,b)$ is a basic open neighborhood of $\overline{(0,0)}$ in X that does not contain any V_i and hence any $U_i \supset V_i$. Therefore $\overline{(0,0)}$ has no countable neighborhood basis in X. The second claim is immediate as every metric space is first countable. □

Claim 4. *The space* $X = \mathrm{HR}$ *is homotopy equivalent to* Y.

Proof. Define

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} & \\ (s,0) & \longmapsto & (0,0) & \\ (s,n) & \longmapsto & (s,n) & \qquad n \geq 1 \end{array}$$

which is continuous. Then $\beta \circ \tilde{f} : \widetilde{X} \to Y$ factors as follows

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f} & Y \end{array}$$

to yield a continuous map $f : X \to Y$ which crushes $\alpha(I_0)$ to the basepoint $\overline{(0,0)}$ of Y.

Define

$$\begin{array}{rcll} \widetilde{Y} & \xrightarrow{\tilde{g}} & X & \\ (0,0) & \longmapsto & \overline{(0,0)} & \\ (s,n) & \longmapsto & \overline{(2s,n)} & 0 \leq s \leq \tfrac{1}{2} \ \& \ n \geq 1 \\ (s,n) & \longmapsto & \overline{\left(\tfrac{2-2s}{n},0\right)} & \tfrac{1}{2} \leq s \leq 1 \ \& \ n \geq 1 \end{array}$$

which is continuous and factors as follows

$$\begin{array}{ccc} & & \widetilde{Y} \\ & \swarrow^{\tilde{g}} & \downarrow{\scriptstyle\beta} \\ X & \xleftarrow[g]{} & Y \end{array}$$

to yield a continuous map $g : Y \to X$ which maps the first half of the nth circle $\beta(I_n)$ of Y onto the nth rope $\alpha(I_n)$ of X and the second half onto $\alpha([0,1/n] \times \{0\})$. We will show that f and g are homotopy inverses by constructing homotopies

$$H : X \times I \to X$$
$$G : Y \times I \to Y$$

such that

(15) $$H_0 = g \circ f \text{ and } H_1 = \mathrm{id}_X$$

(16) $$G_0 = f \circ g \text{ and } G_1 = \mathrm{id}_Y$$

Define

$$\begin{array}{rcl} I_0 \times I & \xrightarrow{H^0} & I_0 \\ (s,0,t) & \longmapsto & (t \cdot s, 0) \end{array}$$

For each $n \geq 1$, let

$$S_n = \{(s,n,t) \mid 0 \leq s \leq (1+t)/2 \ \& \ 0 \leq t \leq 1\} \text{ and}$$
$$T_n = \{(s,n,t) \mid (1+t)/2 \leq s \leq 1 \ \& \ 0 \leq t \leq 1\}.$$

Define

$$\begin{array}{rcl} S_n & \xrightarrow{H^n_-} & I_n \\ (s,n,t) & \longmapsto & \left(\tfrac{2s}{1+t}, n\right) \end{array}$$

and

$$\begin{array}{rcl} T_n & \xrightarrow{H^n_+} & I_0 \\ (s,n,t) & \longmapsto & \left(\tfrac{2+t-2s}{n}, 0\right) \end{array}$$

Define

$$\begin{array}{rcll} \widetilde{X} \times I & \xrightarrow{\widetilde{H}} & X & \\ (x,t) & \longmapsto & \overline{H^0(x,t)} & (x,t) \in I_0 \times I \\ (x,t) & \longmapsto & \overline{H^n_-(x,t)} & (x,t) \in S_n \\ (x,t) & \longmapsto & \overline{H^n_+(x,t)} & (x,t) \in T_n \end{array}$$

which is continuous and factors as follows

$$\begin{array}{ccc} \widetilde{X}\times I & & \\ {\scriptstyle \alpha\times \mathrm{id}_I}\downarrow & \searrow^{\widetilde{H}} & \\ X\times I & \xrightarrow[H]{} & X \end{array}$$

where $\alpha \times \mathrm{id}_I$ is a quotient map since α is a quotient map and I is (locally) compact and Hausdorff. By the universal property H is continuous, and one may verify that (15) holds.

The space X is Hausdorff by Corollary 1 above and Y is Hausdorff by inspection. Therefore

$$\alpha : I_n \to \alpha(I_n) \subset X$$

is a homeomorphism for each $n \geq 0$ and

$$\beta : I_n \to \beta(I_n) \subset Y$$

is a quotient map for each $n \geq 1$.

Define

$$\begin{array}{lcll} \widetilde{Y}\times I & \xrightarrow{\widetilde{G}} & \widetilde{Y} & \\ (0,0,t) & \longmapsto & (0,0) & \\ (s,n,t) & \longmapsto & \left(\frac{2s}{1+t}, n\right) & 0 \leq s \leq \frac{1+t}{2} \ \&\ n \geq 1 \\ (s,n,t) & \longmapsto & (1,n) & \frac{1+t}{2} \leq s \leq 1 \ \&\ n \geq 1 \end{array}$$

which is continuous and factors as follows

$$\begin{array}{ccc} \widetilde{Y}\times I & \xrightarrow{\widetilde{G}} & \widetilde{Y} \\ {\scriptstyle \beta\times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle \beta} \\ Y\times I & \xrightarrow{G} & Y \end{array}$$

where $\beta \times \mathrm{id}_I$ is a quotient map since β is a quotient map and I is (locally) compact and Hausdorff. By the universal property G is continuous, and one may verify that (16) holds. This completes the proof that X and Y are homotopy equivalent. □

The fundamental group $\pi_1\left(Y, \overline{(0,0)}\right)$ is the free group

$$\langle [g_1], [g_2], [g_3], \ldots \rangle$$

where

$$\begin{array}{ccc} I & \xrightarrow{g_n} & Y \\ s & \longmapsto & \overline{(s,n)} \end{array}$$

for each $n \geq 1$. As $g_\sharp$ is an isomorphism, $\pi_1\left(X, \overline{(0,0)}\right)$ is the free group

$$\langle [h_1], [h_2], [h_3], \ldots \rangle$$

where $h_n = g \circ g_n$ for each $n \geq 1$.

Claim 5. *The quotient space X/A is homeomorphic to* HE.

Proof. Recall that $X = \mathrm{HR}$ is depicted in Figure 3, A is the union of the curved arcs (ropes) depicted in Figure 6, and HE is the Hawaiian earring shown in Figure 5. Up to homeomorphism, the radii of the circles in HE are unimportant as long as they tend to zero. To see this, note that any such space is compact and Hausdorff, so the obvious bijection between any two of them is a homeomorphism (see also [7]). For precision we take HE to be

$$Z = \bigcup_{n\geq 1} C\left(1-\frac{1}{2n},\frac{1}{2n}\right)$$

where

$$C(a,r) = \left\{(x,y)\in\mathbb{R}^2 \mid (x-a)^2+y^2 = r^2\right\}.$$

Define $\eta : X \to Z$ by $a \mapsto (1,0)$ for each $a \in A$ and

$$\overline{(s,0)} \mapsto \left(1-\frac{1}{2n},0\right) + \frac{1}{2n}\left(\cos\left(2\pi n\left[(n+1)s-1\right]\right), \sin\left(2\pi n\left[(n+1)s-1\right]\right)\right)$$

for each $n \geq 1$ and $\frac{1}{n+1} \leq s \leq \frac{1}{n}$. Notice that η is a continuous surjection and $\eta^{-1}((1,0)) = A$. Our tactic is to show that η is a quotient map. Once we show this, the universal property of quotient maps will imply that Z is homeomorphic to X/A.

For each $n \geq 1$ define

$$Z_n = C\left(1-\frac{1}{2n},\frac{1}{2n}\right) - \{(1,0)\}$$

which is open in Z. Thus

$$Z - \{(1,0)\} = \coprod_{n\geq 1} Z_n$$

is a topological sum. For each $n \geq 1$ define

$$X_n = \alpha\left(\left(\frac{1}{n+1},\frac{1}{n}\right)\times\{0\}\right)$$

which is open in X. Thus

$$X - A = \coprod_{n\geq 1} X_n$$

is a topological sum and is open in X. Note that

$$\eta| : X - A \to Z - \{(1,0)\} \tag{17}$$

is a continuous bijection. We shall prove that it is a homeomorphism. Since the domain and range are both topological sums it suffices to show that $\eta| : X_n \to Z_n$ is a homeomorphism for each $n \geq 1$. For each $n \geq 1$ we have the closures

$$\mathrm{Cl}\left(X_n\right) = \alpha\left(\left[\frac{1}{n+1},\frac{1}{n}\right]\times\{0\}\right) \text{ and}$$
$$\mathrm{Cl}\left(Z_n\right) = C\left(1-\frac{1}{2n},\frac{1}{2n}\right).$$

Consider the maps

$$\begin{aligned}\left[\tfrac{1}{n+1},\tfrac{1}{n}\right] &\xrightarrow{f_n} \mathrm{Cl}\left(X_n\right)\\ s &\longmapsto \alpha((s,0))\end{aligned}$$

and

$$\left[\tfrac{1}{n+1},\tfrac{1}{n}\right] \xrightarrow{g_n} \mathrm{Cl}\left(Z_n\right)$$

where g_n is given by

$$s \mapsto \left(1-\frac{1}{2n},0\right)+\frac{1}{2n}\left(\cos\left(2\pi n\left[(n+1)s-1\right]\right),\sin\left(2\pi n\left[(n+1)s-1\right]\right)\right).$$

Both are continuous surjections from a compact space to a Hausdorff space and f_n is bijective, so f_n is a homeomorphism and g_n is a quotient map. Notice that

$$f_n^{-1}\left(X_n\right)=\left(\frac{1}{n+1},\frac{1}{n}\right)=g_n^{-1}\left(Z_n\right).$$

It follows that

$$f_n| : \left(\frac{1}{n+1},\frac{1}{n}\right)\to X_n$$

is a homeomorphism and

$$g_n| : \left(\frac{1}{n+1},\frac{1}{n}\right)\to Z_n$$

is a quotient map. But $g_n|$ is bijective hence it is a homeomorphism as well. We have the commutative diagram

$$\begin{array}{ccc} \left(\tfrac{1}{n+1},\tfrac{1}{n}\right) & & \\ {\scriptstyle f_n|}\big\downarrow\approx & \overset{g_n|}{\searrow}\;\approx & \\ X_n & \xrightarrow[\eta|]{} & Z_n \end{array}$$

Therefore $\eta| : X_n \to Z_n$ is a homeomorphism. As noted above, this implies that (17) is a homeomorphism.

We now complete the proof that η is a quotient map. Let V be a subset of Z for which $\eta^{-1}(V)$ is open in X. By (17)

$$\eta| : \eta^{-1}(V)-A\to V-\{(1,0)\}$$

is a homeomorphism from an open subset of $X-A$ onto an open subset of $Z-\{(1,0)\}$. Since $Z-\{(1,0)\}$ is open in Z, it follows that $V-\{(1,0)\}$ is open in Z. In particular, every point of $V-\{(1,0)\}$ is in the interior $\mathrm{Int}(V)$. If $(1,0)\notin V$, we have that V is open in Z and we are done. Otherwise $(1,0)\in V$. In this case $\eta^{-1}(V)$ is an open neighborhood of A. Recall the quotient map $\alpha : \widetilde{X}\to X$. Note that

$$\alpha^{-1}(A)=\{(0,0)\}\cup\{(1/n,0)\mid n\geq 1\}\cup(I\times\mathbb{N})$$

and $\alpha^{-1}\left(\eta^{-1}(V)\right)\subset\widetilde{X}$ is an open neighborhood of $\alpha^{-1}(A)$. It follows that there exist $\epsilon>0$ and sequences $\{a_n\}$ and $\{b_n\}$ such that for $n\geq 1$ we have

$$\frac{1}{n+1}<b_{n+1}<a_n<\frac{1}{n}$$

and

$$([0,\epsilon)\times\{0\})\cup((a_1,1]\times\{0\})\cup\left(\bigcup_{n>1}(a_n,b_n)\times\{0\}\right)\cup(I\times\mathbb{N})\subset\alpha^{-1}\left(\eta^{-1}(V)\right).$$

Let

$$N = \min\{n \geq 1 \mid b_{n+1} < \epsilon\}$$

and let

$$\widetilde{K} = \bigcup_{1 \leq n \leq N} [b_{n+1}, a_n] \times \{0\}.$$

Then $\widetilde{K}$ is a compact subset of $\widetilde{X}$ and

$$\widetilde{X} - \alpha^{-1}\left(\eta^{-1}(V)\right) \subset \widetilde{K}.$$

This implies that

$$X - \eta^{-1}(V) \subset \alpha\left(\widetilde{K}\right)$$

and, hence, that

$$Z - V \subset \eta\left(\alpha\left(\widetilde{K}\right)\right).$$

Now $\eta\left(\alpha\left(\widetilde{K}\right)\right)$ is a compact subset of the Hausdorff space Z and thus is closed in Z. It follows that $Z - \eta\left(\alpha\left(\widetilde{K}\right)\right)$ is contained in V and is open in Z. Note that

$$\widetilde{K} \cap \alpha^{-1}(A) = \varnothing$$

and, hence, that

$$\eta\left(\alpha\left(\widetilde{K}\right)\right) \cap \{(1,0)\} = \varnothing.$$

It follows that

$$(1,0) \in Z - \eta\left(\alpha\left(\widetilde{K}\right)\right) \subset V$$

and so $Z - \eta\left(\alpha\left(\widetilde{K}\right)\right)$ is an open neighborhood of $(1,0)$ contained in V. This implies that $(1,0) \in \mathrm{Int}(V)$. Since $V - \{(1,0)\} \subset \mathrm{Int}(V)$, it follows that $V \subset \mathrm{Int}(V)$. That is to say, V is open in Z. This completes the proof that $\eta : X \to Z$ is a quotient map. Hence X/A is homeomorphic to HE as desired. □

Claim 6. *The space* $X = \mathrm{HR}$ *is locally contractible.*

Proof. The claim is nontrivial only for neighborhoods of the wild point $\overline{(0,0)}$ and there it suffices to show that each basic open set $V(N,a,b)$ from Claim 3 is contractible. So let $V = V(N,a,b)$ be a basic open neighborhood of $\overline{(0,0)}$ in X. We will show that $\overline{(0,0)}$ is a strong deformation retract of V thus completing the proof. Let

$$J = [0, 1/N) \times \{0\},$$
$$U_0 = \bigcup_{n \geq 1} [0, a_n) \times \{n\}, \text{ and}$$
$$U_1 = \bigcup_{n > N} (b_n, 1] \times \{n\}$$

so that

$$\widetilde{V} = \alpha^{-1}(V) = J \cup U_0 \cup U_1.$$

The set $\widetilde{V}$ is open and saturated in $\widetilde{X}$. In particular $\alpha|_{\widetilde{V}} : \widetilde{V} \to V$ is a quotient map. Define

$$\begin{array}{rcll} \widetilde{V} \times I & \xrightarrow{\widetilde{F}} & \widetilde{V} & \\ (s,0,t) & \longmapsto & (s,0) & \\ (s,n,t) & \longmapsto & ((1-t)s,n) & 0 \leq s < \frac{1}{2} \ \&\ n \geq 1 \\ (s,n,t) & \longmapsto & ((1-t)s+t,n) & \frac{1}{2} < s \leq 1 \ \&\ n > N \end{array}$$

which is continuous. Consider the diagram

$$\begin{array}{ccc} \widetilde{V} \times I & \xrightarrow{\widetilde{F}} & \widetilde{V} \\ {\scriptstyle \alpha|_{\widetilde{V}} \times \mathrm{id}_I} \downarrow & & \downarrow {\scriptstyle \alpha} \\ V \times I & \xrightarrow{F} & V \end{array} \tag{18}$$

As $\alpha|_{\widetilde{V}}$ is a quotient map and I is (locally) compact and Hausdorff, $\alpha|_{\widetilde{V}} \times \mathrm{id}_I$ is a quotient map. It is easy to check that $\alpha \circ \widetilde{F}$ is constant on each fiber of $\left(\alpha|_{\widetilde{V}} \times \mathrm{id}_I\right)^{-1}$. So, the universal property implies that the unique function F making (18) commute is continuous. It is simple to check that F is a strong deformation retraction of V to $\alpha(J)$ with $F_0 = \mathrm{id}_V$.

The restriction $\alpha|_J : J \to \alpha(J)$ is a continuous bijection and an open map. Thus $\alpha|_J$ is a homeomorphism onto its image. It is now trivial to construct a strong deformation retraction

$$G : \alpha(J) \times I \to \alpha(J)$$

of $\alpha(J)$ to $\overline{(0,0)}$ with $G_0 = \mathrm{id}_{\alpha(J)}$. Concatenating the homotopies F and G yields the desired strong deformation retraction of V to $\overline{(0,0)}$. □

We now show that $A \subset X$ (see Figure 6) has all trivial homotopy groups (i.e. is weakly contractible) but is not contractible.

Claim 7. *If $K \subset A$ is compact, then K intersects at most finitely many of the sets $\alpha(\mathrm{Int}(I_n))$ where $n \in \mathbb{N}$.*

Proof. Otherwise K contains an infinite, discrete, and closed subset. □

Claim 8. *If $K \subset A$ is connected and contains $\overline{(1,n_1)}$ and $\overline{(1,n_2)}$ for distinct $n_1, n_2 \in \mathbb{N}$, then K contains $\alpha(I_{n_1})$ and $\alpha(I_{n_2})$.*

Proof. This easy verification is left to the reader. □

Claim 9. *If $K \subset A$ is compact and connected, then there exists $N \in \mathbb{N}$ such that*

$$K \subset \bigcup_{1 \leq n \leq N} \alpha(I_n).$$

Proof. This follows immediately from Claims 7 and 8. □

Claim 10. *If $N \in \mathbb{N}$, then*

$$\bigcup_{n > N} \alpha(I_n)$$

is a strong deformation retract of A.

Proof. Fix $N \in \mathbb{N}$. Define

$$\widetilde{P} = \coprod_{1 \le n \le N} I_n = I \times \{1, 2, \ldots, N\}$$

and let $\sim$ be the equivalence relation on $\widetilde{P}$ generated by $(0, n) \sim (0, 1)$ for $n \le N$. The quotient space $P = \widetilde{P}/\!\sim$ is the N–prong. The quotient map is $\gamma : \widetilde{P} \to P$. Define

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{h}} & \widetilde{X} \\ (s, n) & \longmapsto & (s, n) \end{array}$$

which is continuous. The composition $\alpha \circ \tilde{h}$ factors as follows

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{h}} & \widetilde{X} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ P & \xrightarrow{h} & X \end{array}$$

to yield the continuous map h. As h is a continuous injection from a compact space to a Hausdorff one, it is a homeomorphism onto its image. Note that $\mathrm{Im}(h) \subset A \subset X$ is closed in X and in A.

Define

$$\begin{array}{ccc} \widetilde{P} \times I & \xrightarrow{\widetilde{G}} & \widetilde{P} \\ (s, n, t) & \longmapsto & (t \cdot s, n) \end{array}$$

which is continuous and factors as follows

$$\begin{array}{ccc} \widetilde{P} \times I & \xrightarrow{\widetilde{G}} & \widetilde{P} \\ {\scriptstyle\gamma \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle\gamma} \\ P \times I & \xrightarrow{G} & P \end{array}$$

where $\gamma \times \mathrm{id}_I$ is a quotient map since γ is a quotient map and I is (locally) compact and Hausdorff. Thus $\overline{(0, 1)}$ is a strong deformation retract of P by the homotopy G.

The map $H = h \times \mathrm{id}_I : P \times I \to A \times I$ is a continuous bijection from a compact space to a Hausdorff one. Hence H is a homeomorphism onto its image

$$\left(\bigcup_{1 \le n \le N} \alpha\left(I_n\right) \right) \times I \subset A \times I$$

which is closed in $A \times I$. Define F^- to be the composition

$$\mathrm{Im}(H) \xrightarrow{H^{-1}} P \times I \xrightarrow{G} P \xrightarrow{h} \mathrm{Im}(h).$$

Note that F_1^- is the identity on $\mathrm{Im}(h)$, F_0^- sends everything to $\overline{(0, 1)}$, and F_t^- fixes $\overline{(0, 1)}$ for every $t \in I$.

The set

$$\bigcup_{n>N} \alpha(I_n) \subset A$$

is closed in X and in A. Further

$$\left(\bigcup_{n>N} \alpha(I_n)\right) \times I \subset A \times I$$

is closed in $A \times I$. Define

$$F^+ : \left(\bigcup_{n>N} \alpha(I_n)\right) \times I \to \bigcup_{n>N} \alpha(I_n)$$

by $(x,t) \mapsto x$ for every $t \in I$. Being a projection, F^+ is continuous. The continuous maps F^- and F^+ are defined on closed subsets of $A \times I$ and they agree on the intersection of their domains, namely $\left\{\overline{(0,1)}\right\} \times I$ where each is projection onto the first factor. Thus these functions paste together to yield the desired homotopy $F : A \times I \to A$. □

Claim 11. *If $k \geq 0$, then $\pi_k(A)$ is trivial.*

Proof. Fix $k \geq 0$. Let $f : S^k \to A$ be continuous. It suffices to show that f is homotopic to a constant map. There exists $N \in \mathbb{N}$ such that

$$\mathrm{Im}(f) \subset \bigcup_{1 \leq n \leq N} \alpha(I_n).$$

For $k = 0$ this is obvious and for $k \geq 1$ it follows from Claim 9. Let $F : A \times I \to A$ be the homotopy provided by Claim 10. The map

$$f \times \mathrm{id}_I : S^k \times I \to A \times I$$

is continuous. The composition $F \circ (f \times \mathrm{id}_I)$ is the desired homotopy. □

The following claim immediately implies that A is not contractible.

Claim 12. *If $F : A \times I \to A$ is a homotopy with $F_0 = \mathrm{id}_A$, then there exists a cofinite subset $E \subset \mathbb{N}$ such that*

$$F\left(\overline{(0,1)}, t\right) = \overline{(0,1)} \text{ and} \tag{19}$$

$$F\left(\overline{(1,n)}, t\right) = \overline{(1,n)} \tag{20}$$

for every $n \in E$ and $t \in I$.

Proof. Consider the set $J \subset I$ defined by: $\tau \in J$ if and only if there exists a cofinite subset $E \subset \mathbb{N}$ such that (19) and (20) hold for every $n \in E$ and $t \in [0, \tau]$. As $F_0 = \mathrm{id}_A$, $0 \in J$. Let b be the least upper bound of J. Then J is a connected interval containing $[0, b)$. First we show that $b \in J$.

For each $a \in A$ the set $\{t \in I \mid F(a,t) = a\}$ is closed in I since A is Hausdorff. Thus (19) holds for $t \in [0, b]$. Let

$$U = A - \left\{\overline{(1/2, n)} \mid n \in \mathbb{N}\right\}$$

which is just the ropes A (see Figure 6) minus their midpoints. As U is open in A, $F^{-1}(U)$ is open in $A \times I$. Also

$$\overline{(0,1)} \times [0,b] \subset F^{-1}(U).$$

By compactness of $[0,b]$, there is an open set $V \subset A$ so that the tube $V \times [0,b] \subset F^{-1}(U)$ contains $\overline{(0,1)} \times [0,b]$. As V is an open neighborhood of $\overline{(0,1)}$ in A and $A \subset X$ has the subspace topology, there exists a cofinite subset $D \subset \mathbb{N}$ such that $\overline{(1,n)} \in V$ for $n \in D$. Therefore

$$F\left(\overline{(1,n)}, t\right) \in U$$

for every $n \in D$ and $t \in [0,b]$. By a connectedness argument we get

$$F\left(\overline{(1,n)}, t\right) \in \alpha\left((1/2,1] \times \{n\}\right)$$

for every $n \in D$ and $t \in [0,b]$. We now show there exists a cofinite subset $E \subset D$ such that (20) holds for every $n \in E$ and $t \in [0,b]$. Suppose not, then there is a sequence $1 \le n_1 < n_2 < \cdots$ of numbers in D and a sequence $t_k \in [0,b]$, $k \in \mathbb{N}$, such that

$$F\left(\overline{(1,n_k)}, t_k\right) \in \alpha\left((1/2,1) \times \{n_k\}\right) \tag{21}$$

for each $k \in \mathbb{N}$. Without loss of generality, we assume the sequence $\{t_k\}$ converges since it is bounded and has a convergent subsequence. Note that the sequence $\left\{\overline{(1,n_k)}\right\}$ converges to $\overline{(0,1)}$ in A. Thus the sequence $\left\{\left(\overline{(1,n_k)}, t_k\right)\right\}$ converges in $A \times I$, however by (21) its image under the continuous function $F : A \times I \to A$ does not converge which is a contradiction. Thus the cofinite subset $E \subset D$ exists. Hence $b \in J$.

Thus $J = [0,b]$. Suppose $b < 1$. Note that $F^{-1}(U)$ is an open neighborhood of $\overline{(0,1)} \times [0,b]$ in $A \times I$. As above, there is a tube in $F^{-1}(U)$ containing $\overline{(0,1)} \times [0,b]$. This tube combines with an open rectangle neighborhood of $\left(\overline{(0,1)}, b\right)$ in $F^{-1}(U)$ to yield an open set $V \subset A$ and an $\epsilon > 0$ so that

$$\overline{(0,1)} \times [0, b+\epsilon) \subset V \times [0, b+\epsilon) \subset F^{-1}(U).$$

There is a cofinite subset $D \subset \mathbb{N}$ such that $\overline{(1,n)} \in V$ for $n \in D$. Using connectedness as above, we get

$$F\left(\overline{(1,n)}, t\right) \in \alpha\left((1/2,1] \times \{n\}\right)$$

for every $n \in D$ and $t \in [0, b+\epsilon)$. We now show there exists a cofinite subset $E \subset D$ such that (20) holds for every $n \in E$ and $t \in [0, b+\epsilon)$. Suppose not, then there is a sequence $1 \le n_1 < n_2 < \cdots$ of numbers in D and a sequence $t_k \in [0, b+\epsilon)$, $k \in \mathbb{N}$, such that (21) holds for each $k \in \mathbb{N}$. Without loss of generality, the sequence $\{t_k\}$ converges. Thus the sequence $\left\{\left(\overline{(1,n_k)}, t_k\right)\right\}$ converges in $A \times I$, however by (21) its image under the continuous function $F : A \times I \to A$ does not converge which is a contradiction. Thus the cofinite subset $E \subset D$ exists. Therefore (20) holds for every $n \in E$ and $t \in [0, b+\epsilon)$. Fix $t \in [0, b+\epsilon)$. Then

$$\left(\overline{(1,n)}, t\right) \to \left(\overline{(0,1)}, t\right)$$

in $A \times I$. As F is continuous we have

$$F\left(\overline{(1,n)}, t\right) \to F\left(\overline{(0,1)}, t\right)$$

in A. As $E \subset \mathbb{N}$ is cofinite and (20) holds for $n \in E$ and our fixed t, we have

$$F\left(\overline{(1,n)}, t\right) \to \overline{(0,1)}.$$

As A is Hausdorff, limits are unique. Therefore (19) holds for our fixed t. So (19) and (20) hold for every $n \in E$ and $t \in [0, b+\epsilon)$. This contradicts the assumption that b is the least upper bound of J. Therefore $b = 1$, $J = [0,1]$, and the claim is proved. □

We close this example by showing X is paracompact. As X is Hausdorff (see Corollary 1 above), the following claim and [6, Thm. 41.1] imply that X is a normal space.

Claim 13. *The space* $X = \mathrm{HR}$ *is paracompact.*

Proof. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover of X where J is an arbitrary index set. We must produce a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ that covers X. As $\mathcal{U}$ is an open cover of X there exists $U_0 \in \mathcal{U}$ with $\overline{(0,0)} \in U_0$. By Claim 3 there is a basic open neighborhood $V = V(N,a,b)$ of $\overline{(0,0)}$ contained in U_0. Figure 7 depicts $\widetilde{V} = \alpha^{-1}(V)$.

Fix an arbitrary $n > N$. Observe that $\alpha\left([a_n, b_n] \times \{n\}\right)$ is compact and thus is covered by finitely many open sets

$$U(n,1), U(n,2), \ldots, U(n, M_n) \in \mathcal{U}$$

for some $M_n \in \mathbb{N}$. The interval

$$\left(\frac{a_n}{2}, \frac{1+b_n}{2}\right) \times \{n\} \subset I_n$$

is an open and saturated subset of $\widetilde{X}$. Thus

$$\alpha\left(\left(\frac{a_n}{2}, \frac{1+b_n}{2}\right) \times \{n\}\right)$$

is open in X, is contained within the nth rope of X, and contains $\alpha\left([a_n, b_n] \times \{n\}\right)$. For each $1 \le k \le M_n$ define

$$V(n,k) = U(n,k) \cap \alpha\left(\left(\frac{a_n}{2}, \frac{1+b_n}{2}\right) \times \{n\}\right)$$

which is open in X. By construction, each $V(n,k)$ is contained in some element of $\mathcal{U}$ and

$$\alpha\left([a_n, b_n] \times \{n\}\right) \subset \left(\bigcup_{1 \le k \le M_n} V(n,k)\right) \subset \alpha\left(\left(\frac{a_n}{2}, \frac{1+b_n}{2}\right) \times \{n\}\right). \tag{22}$$

Next let $a_0 = 1/N$ and let

$$\widetilde{K} = (I \times \{0,1,2,\ldots,N\}) - \widetilde{V} = \bigcup_{0 \le n \le N} [a_n, 1] \times \{n\}$$

which is compact. Thus $K = \alpha\left(\widetilde{K}\right) \subset X$ is compact and is covered by finitely many open sets

$$V_1, V_2, \dots, V_M \in \mathcal{U}$$

for some $M \in \mathbb{N}$. Define

$$\mathcal{V} = \{V, V_1, V_2, \dots, V_M\} \cup \{V(n,k) \mid n > N \ \& \ 1 \leq k \leq M_n\}$$

which, by construction, is an open cover of X that refines the cover $\mathcal{U}$. It remains to show that $\mathcal{V}$ is locally finite. For each $x \in X$ we must produce an open neighborhood $W_x \subset X$ of x that intersects nontrivially only finitely many elements in $\mathcal{V}$. First, if $x \in \alpha((0,1) \times \{n\})$ for some $n \geq 1$, then let

$$W_x = \alpha((0,1) \times \{n\})$$

which is an open neighborhood of x in X. Now, if $1 \leq n \leq N$, then W_x intersects at most $V, V_1, \dots, V_M$, while if $n > N$, then W_x may further intersect at most $V(n,1), \dots, V(n, M_n)$. Second, if $x = \overline{(0,0)}$, then recall $V = V(N,a,b)$ from earlier in the proof and define the sequences a' and b' by $a'_n = a_n/2$ and $b'_n = (1+b_n)/2$. Let

$$W_x = V(N, a', b')$$

which is a basic open neighborhood of $x = \overline{(0,0)}$ in X. By (22), W_x intersects at most $V, V_1, \dots, V_M$. Third, if $\alpha^{-1}(x) = \{(s,0)\}$ is a single point in I_0 where $1/(n+1) < s < 1/n$ for some $n \in \mathbb{N}$, then let

$$W_x = \alpha\left(\left(\frac{1}{n+1}, \frac{1}{n}\right) \times \{0\}\right)$$

which is an open neighborhood of x in X that intersects at most $V, V_1, \dots, V_M$. Fourth, if $\alpha^{-1}(x) = \{(1/n, 0), (1,n)\}$, then let

$$W_x = \alpha\left(\left(\frac{1}{n+1}, \frac{1}{n-1}\right) \times \{0\}\right) \cup \alpha\left(\left(\frac{1}{2}, 1\right] \times \{n\}\right)$$

when $n > 1$ and let

$$W_x = \alpha\left(\left(\frac{1}{2}, 1\right] \times \{0\}\right) \cup \alpha\left(\left(\frac{1}{2}, 1\right] \times \{n\}\right)$$

when $n = 1$. In both subcases W_x is an open neighborhood of x in X. Now, if $1 \leq n \leq N$, then W_x intersects at most $V, V_1, \dots, V_M$, while if $n > N$, then W_x may further intersect at most $V(n,1), \dots, V(n, M_n)$. These four cases exhaust all possibilities and the claim follows. □

3.5. **Semi-local simple connectedness II.** The previous example raises the question of whether nontriviality of $\pi_1(X)$ is necessary in order to create new π_1 in the quotient $X/\!\sim$. This condition is not necessary, as we now show. Let X be the cone on HR as follows

$$\mathrm{HR} \times I \xrightarrow{q} \mathrm{HR} \times I / \mathrm{HR} \times \{0\} = X.$$

Recall the closed subspace $A \subset \mathrm{HR}$ depicted in Figure 6 and defined in equation (11) above. Define the quotient map

$$X \xrightarrow{\pi} X / q(A \times \{1\}).$$

Claim 14. *The space $X/q(A \times \{1\})$ is given as the quotient of a contractible and locally contractible space by a closed weakly contractible subspace.*

Proof. As X is the cone on the locally contractible space HR (see Claim 6), it is contractible and locally contractible. Observe that $q|_{\mathrm{HR}\times(0,1]}$ is a homeomorphism onto its image. To see this, note that $q|_{\mathrm{HR}\times(0,1]}$ is clearly a continuous bijection and, being the restriction of a quotient map to an open saturated subset, is itself a quotient map. Thus $q|_{\mathrm{HR}\times(0,1]}$ is an open map and hence a homeomorphism. In particular $q(A\times\{1\})$ is homeomorphic to A which is weakly contractible by Lemma 11. Lastly $q(A\times\{1\})$ is closed in $q(\mathrm{HR}\times\{1\})$ which in turn is closed in X and so $q(A\times\{1\})$ is closed in X. □

Claim 15. *The space X is Hausdorff and normal but is not metrizable.*

Proof. By Corollary 1, HR is Hausdorff. Thus $\mathrm{HR}\times I$ and the cone X on HR are both Hausdorff spaces. By Claim 13, HR is paracompact and normal, thus a result of Dowker [3] implies that $\mathrm{HR}\times I$ is normal. An easy exercise shows that a normal space mod a closed subspace is normal. As $\mathrm{HR}\times\{0\}$ is closed in $\mathrm{HR}\times I$, we see that X is normal. Alternatively one may see that normality of $\mathrm{HR}\times I$ implies X is normal by realizing the cone X as an obvious adjunction space of $\mathrm{HR}\times I$ and the one point space $\{*\}$ and then applying [5, Thm. 37.2]. Nonmetrizability of X is immediate: metrizability is hereditary, HR is homeomorphic to $\mathrm{HR}\times\{1\}\subset X$, and HR is not metrizable by Corollary 2 in the previous section. □

We complete this example by showing $X/q(A\times\{1\})$ has an uncountable first homology group and hence an uncountable (and nontrivial!) fundamental group. All homology groups are taken with integer coefficients.

Claim 16. *The group $H_1(X/q(A\times\{1\}))$ is uncountable.*

Proof. Define the closed subspaces $T,B\subset X/q(A\times\{1\})$ (for top and bottom) as follows

$$T=\pi\circ q(\mathrm{HR}\times[0,2/3]) \text{ and}$$
$$B=\pi\circ q(\mathrm{HR}\times[1/3,1]).$$

Clearly T is contractible (being a cone). Observe that $\pi\circ q|_{\mathrm{HR}\times(0,1)}$ is a homeomorphism onto its image. To see this, note that $\pi\circ q$ is a quotient map since it is the composition of quotient maps. Now reason as in the proof of Claim 14. So $T\cap B$ is homeomorphic to $\mathrm{HR}\times[1/3,2/3]$ and hence is homotopy equivalent to HR. Let f be the quotient map

$$\mathrm{HR}\xrightarrow{f}\mathrm{HR}/A$$

Notice that B is homeomorphic to the mapping cylinder of f. Thus B is homotopy equivalent to HE by Claim 5 above. Applying the Mayer-Vietoris sequence to the splitting of $X/q(A\times\{1\})$ into $T\cup B$ we see that

$$H_1(X/q(A\times\{1\}))\cong H_1(\mathrm{HE})/\mathrm{Im}(H_1(\mathrm{HR})).$$

Now $\pi_1(\mathrm{HR})$ is free on a countably infinite number of generators, as shown in the previous section, and is therefore a countable group. Thus its quotient $H_1(\mathrm{HR})$ is countable as well. So it suffices to show that $H_1(\mathrm{HE})$ is uncountable. For each $n\in\mathbb{N}$ the winding number around the nth circle $C_n\subset\mathrm{HE}$ is well-defined on $H_1(\mathrm{HE})$ as shown using the obvious retraction $\mathrm{HE}\to C_n$. Using this observation it is straightforward to construct for each S in the power set of $\mathbb{N}$ a singular 1–cycle in HE (with a single 1–simplex) whose winding number around C_n is nonzero if and only if $n \in S$. Thus H_1 (HE) is uncountable as desired. □

3.6. **Stereographic projection of Hopf field.** Multiplication by $i \in \mathbb{C}$ on $\mathbb{C}^n$ is the vector field given in real coordinates by

$$v(x) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1})^t .$$

Clearly v is tangent to each sphere about the origin in $\mathbb{R}^{2n}$. Let $S^{2n-1} \subset \mathbb{R}^{2n}$ denote the unit sphere. The orbits of v on S^{2n-1} are great circles, one for each complex line through $0 \in \mathbb{C}^n$. The orbit space of v restricted to S^{2n-1} is $\mathbb{C}P^{n-1} \cong S^{2n-1}/S^1$.

We obtain the vector field u on $\mathbb{R}^{2n-1} = \left\{y \in \mathbb{R}^{2n} \mid y_{2n} = 0\right\}$ using stereographic projection as follows. Fix $p = (0, 0, \ldots, 0, 1) \in S^{2n-1}$ and let $x = (x_1, x_2, \ldots, x_{2n})$, then stereographic projection

$$s : S^{2n-1} - \{p\} \to \mathbb{R}^{2n-1}$$

is the diffeomorphism given by

$$s(x) = \frac{1}{1 - x_{2n}}(x_1, x_2, \ldots, x_{2n-1}, 0).$$

If $y = (y_1, y_2, \ldots, y_{2n-1}, 0)$, then the inverse diffeomorphism is given by

$$s^{-1}(y) = \frac{2}{1 + \|y\|^2} \left(y_1, y_2, \ldots, y_{2n-1}, \frac{\|y\|^2 - 1}{2} \right).$$

Define u on $\mathbb{R}^{2n-1}$ by applying ds to v. In y coordinates, u is

$$u(y) = ds|_{s^{-1}(y)}\, v\left(s^{-1}(y)\right).$$

Notice that

$$ds|_x = \frac{1}{1 - x_{2n}} [I \mid A]$$

where I is the $(2n-1) \times (2n-1)$ identity matrix and A is the column vector

$$A = \frac{1}{1 - x_{2n}}(x_1, x_2, \ldots, x_{2n-1})^t.$$

So

$$ds|_{s^{-1}(y)} = \frac{1 + \|y\|^2}{2} [I \mid B]$$

where $B = (y_1, y_2, \ldots, y_{2n-1})^t$. As

$$v(s^{-1}(y)) = \frac{2}{1 + \|y\|^2} \left(-y_2, y_1, \ldots, -y_{2n-2}, y_{2n-3}, \frac{1 - \|y\|^2}{2}, y_{2n-1} \right)^t,$$

we have

$$u(y) = \begin{pmatrix} -y_2 + y_1 y_{2n-1} \\ y_1 + y_2 y_{2n-1} \\ -y_4 + y_3 y_{2n-1} \\ y_3 + y_4 y_{2n-1} \\ \vdots \\ -y_{2n-2} + y_{2n-3} y_{2n-1} \\ y_{2n-3} + y_{2n-2} y_{2n-1} \\ \frac{1 - \|y\|^2}{2} + y_{2n-1}^2 \end{pmatrix}.$$

Thus, u is a quadratic polynomial vector field on $\mathbb{R}^{2n-1}$. One orbit of u is a properly embedded copy of $\mathbb{R}$ while the rest are smooth circles. It remains to show that the orbit space $\mathbb{R}^{2n-1}/\!\sim$ with $\sim$ induced by u is homeomorphic to $\mathbb{C}P^{n-1}$.

Consider the diagram

$$\begin{array}{ccc} \mathbb{R}^{2n-1} & \xrightarrow{\;s^{-1}\;} & S^{2n-1} \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathbb{R}^{2n-1}/\!\sim & \overset{f}{\dashrightarrow} & S^{2n-1}/S^1 \end{array}$$

where q and q' are the associated quotient maps. By construction, $q \circ s^{-1}$ is constant on each fiber of q' so the universal property of quotient maps implies that there exists a unique continuous function f making the diagram commute. Also by construction, f is a bijection. Let $D = \{x \in S^{2n-1} \mid x_{2n} \leq 0\}$ be the lower hemisphere of S^{2n-1}. Notice that each orbit in S^{2n-1} intersects D in at least a nontrivial arc of points and that D is compact. Thus $s(D)$ is compact and maps by q' surjectively to $\mathbb{R}^{2n-1}/\!\sim$. Thus $\mathbb{R}^{2n-1}/\!\sim$ is compact. As S^{2n-1}/S^1 is Hausdorff, we have that f is a homeomorphism as desired.

3.7. **Higher homotopy.** We construct a contractible space partitioned into contractible fibers and satisfying the hypotheses of the main lemma (with all spaces locally contractible), whose quotient, while necessarily simply connected, has higher homotopy and homology. This shows that the analogy of our main lemma with the long exact sequence associated to a fibration does not extend to higher homotopy groups and thus is a special property of the fundamental group.

Fix notation as in the previous section with $n = 2$, so q is the classical Hopf fibration. Let $\sim$ denote the associated equivalence relation on S^3. The unique complex line contained in $\{(x_1 + ix_2, x_3 + ix_4) \mid x_4 = 0\}$ is $\mathbb{C} \times \{0\}$. Thus D intersects orbits of q in the following way: one intersection is a complete orbit $\{(x_1, x_2, 0, 0) \mid x_1^2 + x_2^2 = 1\}$ and the rest are semicircles each intersecting ∂D in a pair of antipodal points. Let $D' = D - \{(1, 0, 0, 0)\}$ and let $\sim$ denote the restriction of $\sim$ to D' where no confusion should arise. We have the diagram

$$\begin{array}{ccc} D' & \xrightarrow{\;j\;} & S^3 \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ D'/\!\sim & \overset{g}{\dashrightarrow} & S^3/S^1 \end{array}$$

where j is inclusion and π is a quotient map. The composition $q \circ j$ is constant on each fiber of π and so the universal property of quotient maps implies that the unique function g making the diagram commute is continuous. It is easy to see that g is a bijection. We claim that $D'/\!\sim$ is compact. To see this let $B \subset \mathbb{R}^4$ denote the open ball of radius $1/2$ centered at $(1, 0, 0, 0)$. Note that $D' - B$ is a compact subset of D'. Note also that each fiber of π has diameter 2 and thus intersects $D' - B$ nontrivially. Therefore $\pi|_{D'-B}$ is surjective and $D'/\!\sim$ is compact. As S^3/S^1 is Hausdorff, g is a homeomorphism. Thus we have D', which is diffeomorphic to closed upper 3–space, partitioned into connected arcs (one open and the rest closed) with quotient S^2, completing the example. Note that stereographic projection from $(1, 0, 0, 0)$ explicitly exhibits a diffeomorphism from D' to half-space, with the partition induced by a quadratic vector field (cf. 3.6), so the example is remarkably simple from an analytic viewpoint.

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