Powers of Gaussian Integers

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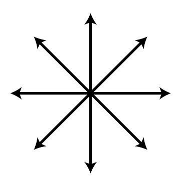
Choose your favorite Gaussian integer $z = a + bi$. Look at the powers z, z^2, z^3, z^4, and so forth. How do they behave geometrically? For most choices of z it's obvious that these powers spiral out away from the origin. Can we say more?

Write $z = re^{i\theta}$ with $r = |z|$ and $\theta = \arg z$. Then $z^n = r^n e^{in\theta}$. The r^n factor is easily understood. Thus, the question comes down to understanding the behavior of the points $e^{in\theta}$, $n \in \mathbb{N}$, on the unit circle.

Recall the following basic consequence of unique factorization of Gaussian integers.

Lemma [1, p. 6]. *Let $z \neq 0$ be a Gaussian integer. There is a natural number n such that z^n is real if and only if z lies on one of the four lines* $\mathrm{Im}z = 0$, $\mathrm{Re}z = \mathrm{Im}z$, $\mathrm{Re}z = 0$, *or* $\mathrm{Re}z = -\mathrm{Im}z$ *in* $\mathbb{C}$ *shown below.*



Corollary. *Let $z \neq 0$ be a Gaussian integer. If* $\arg z$ *is a rational multiple of* π*, then* $\arg z = k\pi/4$ *for some integer k.*

Proof. Suppose $\arg z = m\pi/n$ where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then $\arg z^n = n \arg z = m\pi$ and so z^n is real. Now apply the previous lemma. □

Next, recall the following important theorem of Weyl (also due to Sierpinski, Dirichlet and Oresme in other forms [3, pp. 156–158]).

Theorem. *Let $\alpha \in \mathbb{R}$ be irrational. The set $\{k\alpha\}$, $k \in \mathbb{N}$, is uniformly distributed* $\bmod 1$. *In particular, this set is dense in* $[0, 1)$.

The density part follows from the pigeonhole principle and the uniform distribution part has several different proofs.

The corollary and the theorem above combine to give the following answer to our question. First, if z lies on one of the four lines pictured above, then its powers all lie on those lines as well. More interesting, if z does not lie on one of the four lines, then $\theta = \arg z$ is an irrational multiple of π and so the points $e^{in\theta}$, $n \in \mathbb{N}$, form a uniformly distributed and therefore dense subset of the unit circle. So, even though z^n never lands on the real axis, it comes arbitrarily close in argument.

After we made the above observation we found that a similar observation was made by J. C. Lagarias [2].

References

[1] J. S. Calcut, Gaussian integers and arctangent identities for pi, *American Math. Monthly* accepted for publication 1–20.

[2] J. C. Lagarias, Density of arguments of powers of Gaussian integers, *American Math. Monthly* **84** (1977) 493.

[3] Karl Petersen, *Ergodic theory*, Cambridge University Press, 1990.

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