Ideal Pendulum Handout #3
Math 427K
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Recall: start the pendulum at $\pi/2 \leq \theta(0) < \pi$ and let T denote the time it takes the pendulum to fall to $\pi/2$ (horizontal). We found a lower bound for T:

$$LB = \sqrt{L/g}\cosh^{-1}\left(\frac{\pi}{2\left[\pi - \theta(0)\right]}\right)$$

and we found an upper bound for T:

$$UB = \sqrt{\pi/2}\sqrt{L/g}\cosh^{-1}\left(\frac{\pi}{2\left[\pi - \theta(0)\right]}\right).$$

Note that $UB = \sqrt{\pi/2}LB \approx 1.2533LB$. Thus we have:

$$LB \leq T \leq \sqrt{\pi/2}LB.$$

These are crude, yet useful bounds.

Example 1 *Let* $L = 1$ m *and* $g = +9.8$ m/s^2 *as in the previous handout. We have:*

$\theta(0)$	*Lower Bound* LB	*Actual Time* T	*Upper bound* UB
$3\pi/6$	0.000	0.000	0.000
$4\pi/6$	0.307	0.344	0.385
$5\pi/6$	0.563	0.590	0.706
$11\pi/12$	0.792	0.811	0.992
$23\pi/24$	1.015	1.033	1.272

Exercise 2 *Why is the actual time* T *closer to the lower bound* LB*?*

Example 3 *Let* $L = 1$ m *and* $g = +9.8$ m/s^2*. Start the pendulum at* $\theta(0) = 179°$*. How long will it take the pendulum to fall to the horizontal position?*
Answer: Between 1.6588 *and* 2.0791 *seconds.*

Exercise 4 *Where should one start the pendulum so that it takes at least* 5 *seconds to fall to the horizontal position? (Besides* $\theta(0) = \pi$*.)*

Damped Motion

The original pendulum ODE:

$$\theta'' = -\frac{g}{L} \sin \theta. \tag{1}$$

ignored friction and wind resistance.

How do we account for these factors?

The following is the simplest and most common method.

We make two observations about friction and wind resistance:

1. They decelerate the object in the direction of motion.
2. Their effects are stronger at higher velocities.

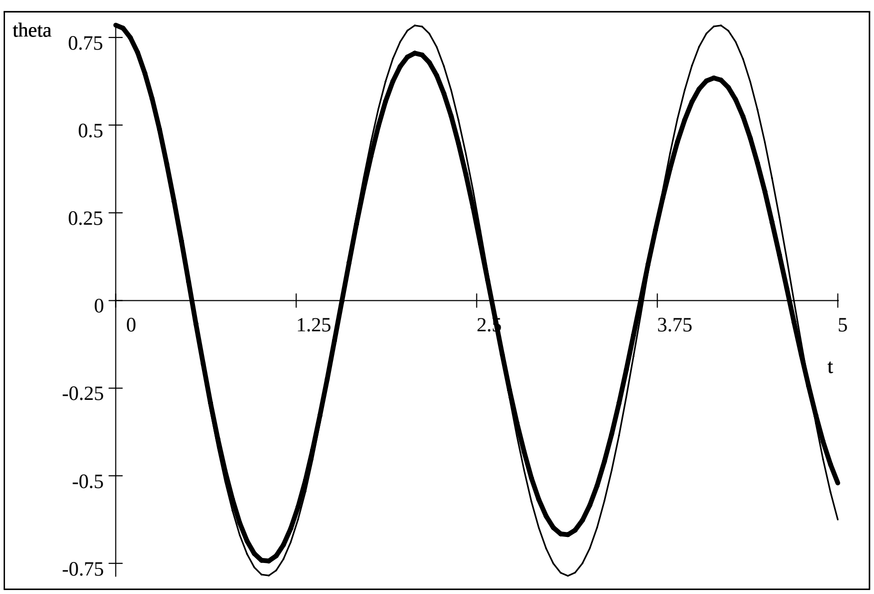
We conclude that acceleration should be reduced by a quantity proportional to velocity.

Thus, we have the ODE of the damped pendulum:

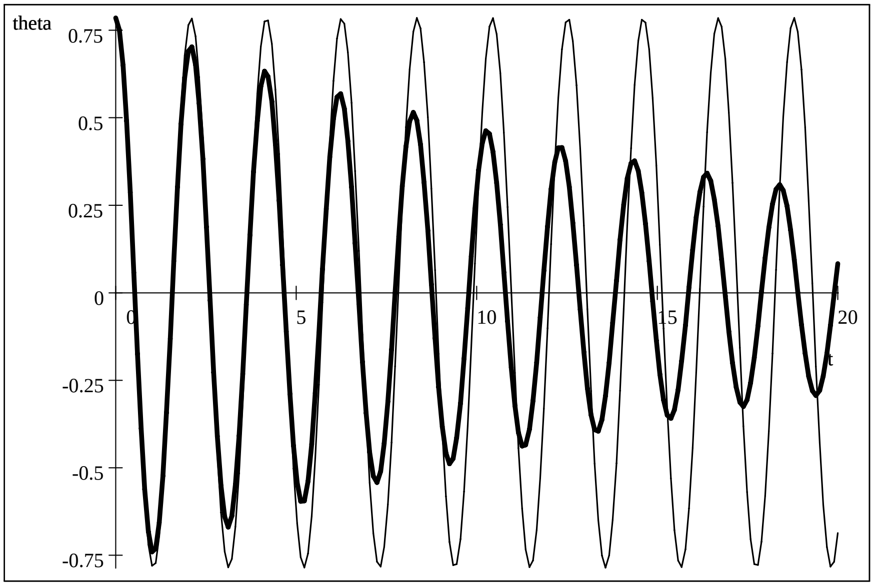
$$\theta'' = -\frac{g}{L} \sin \theta - \gamma \theta', \tag{2}$$

where $\gamma > 0$ is a constant called the *damping coefficient* (γ has units s^{-1}). Usually γ is small.

Example 5 *Let* $L = 1$ m *and* $g = +9.8$ m/s^2. *Let* $\theta(0) = \pi/4$ rad. *We plot* $\theta(t)$ *for undamped and damped motion with damping coefficient* $\gamma = 0.1$ s^{-1}.



Undamped (thin) and damped (thick) pendulum motion.



Undamped (thin) and damped (thick) pendulum motion.

Now, let us linearize (2) at $\theta = 0$ and solve. Again, we replace $\sin\theta$ with θ to obtain:

$$\begin{aligned} \theta'' &= -\frac{g}{L}\theta - \gamma\theta', \text{ or} \\ \theta'' + \gamma\theta' + \frac{g}{L}\theta &= 0. \end{aligned} \tag{3}$$

The characteristic equation is:

$$r^2 + \gamma r + \frac{g}{L} = 0$$

with roots:

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4g/L}}{2}.$$

Usually $\gamma > 0$ is small. So, from here on assume $\gamma^2 - 4g/L < 0$. Thus, r_1 and r_2 are complex conjugates. Write:

$$\begin{aligned} r_1 &= \frac{-\gamma}{2} + i\frac{\sqrt{-\gamma^2 + 4g/L}}{2} \\ &= \lambda + i\mu \end{aligned}$$

where:

$$\lambda = \frac{-\gamma}{2} \text{ and } \mu = \frac{\sqrt{-\gamma^2 + 4g/L}}{2}.$$

Recall that the solution to (3) is:

$$\theta(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t). \tag{4}$$

Key observation: $\lambda < 0$ and so the $e^{\lambda t}$ terms cause $\theta(t)$ to approach zero as t tends to $+\infty$! The damping kills the motion.

Let us rewrite the solution (4). Recall the cosine angle addition formula:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Hence:

$$\begin{aligned}
\theta(t) &= c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \\
\theta(t) &= e^{\lambda t}\left(c_1\cos(\mu t) + c_2\sin(\mu t)\right) \\
\theta(t) &= c e^{\lambda t}\cos(\mu t + \phi)
\end{aligned}$$

where ϕ (the *phase angle*) and c are constants.

Concluding Remarks

We have used the problem of pendulum motion as a guide to explore:

- Producing an ODE that models a physical situation.
- Nonlinear ODEs such as:
$$\theta'' = -\frac{g}{L}\sin\theta.$$
- Linearizing nonlinear ODEs:
$$\theta'' = -\frac{g}{L}\theta \text{ at } \theta = 0.$$
- Understanding the behavior of solutions to ODEs that we cannot explicitly solve.
- Using the complex numbers to obtain real solutions.
- Differences between homogenous and nonhomogeneous equations, specifically in the form of their solutions. See table on page 1 of the second pendulum handout.
- Hyperbolic trigonometric functions.
- Equilibrium solutions
- Stability/Instability of equilibrium solutions.
- Galileo's observation on the period of a pendulum (close, but incorrect).
- Damped motion.

Remark 6 *The pendulum is one of the simplest physical situations, yet working with its ODE is already nontrivial (its solutions are not elementary functions). What should we expect with more complicated physical situations (e.g. spinning top, many body problems, and fluid flow)?*

Exercise 7 *List a few physical situations whose governing ODEs are easier to solve than that of the pendulum.*