Ideal Pendulum Handout #2
Math 427K
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This handout explores examples following the previous handout. Some solutions in this handout are numerical; you are required to understand the results.

The ODE governing the motion of the ideal pendulum is:

$$\theta'' = -\frac{g}{L}\sin\theta. \tag{1}$$

We assume $\theta'(0) = 0$ and $-\pi < \theta(0) \leq \pi$. The two equilibrium solutions to (1) are:

$$\begin{aligned} \theta_0(t) &= 0 \text{ (hang straight down; stable), and} \\ \theta_\pi(t) &= \pi \text{ (point straight up; unstable).} \end{aligned}$$

Throughout we fix $L = 1$ m and $g = +9.8$ m/s^2. This corresponds roughly to a grandfather clock sized pendulum on the surface of the earth.

The linearization of our main ODE (1) is:

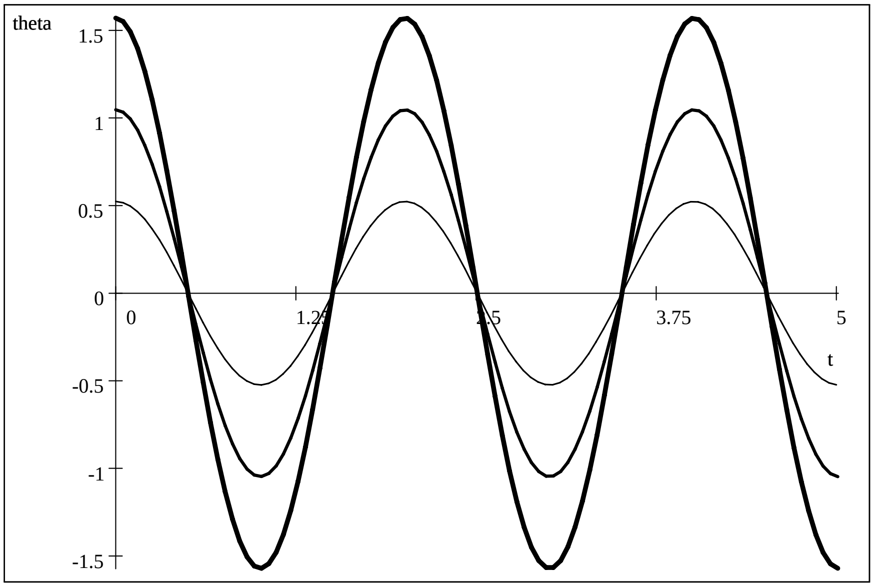
	Linearization	Solution
$\theta = 0$	$\theta'' = -\frac{g}{L}\theta$	$\theta(t) = \theta(0)\cos\left(t\sqrt{g/L}\right)$
$\theta = \pi$	$\theta'' = -\frac{g}{L}(-\theta + \pi)$	$\theta(t) = [\theta(0) - \pi]\cosh\left(t\sqrt{g/L}\right) + \pi$

Recall that the hyperbolic cosine is:

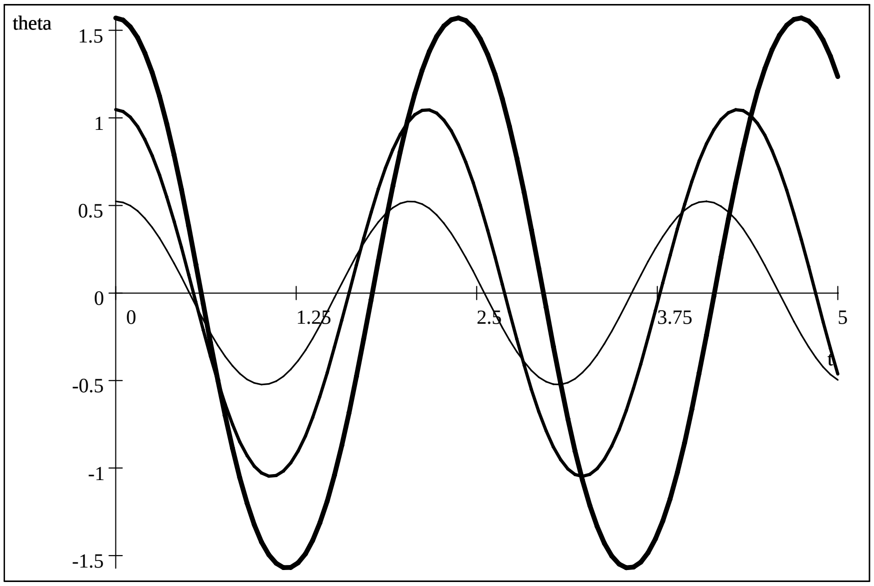
$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Most often in basic physics one studies the linearized ODE at $\theta = 0$ with solution $\theta(t) = \theta(0)\cos\left(t\sqrt{g/L}\right)$ where $\theta(0)$ is the *amplitude* and $2\pi\sqrt{L/g}$ is the *period*. Note: in this case the period is independent of m (mass of pendulum) and $\theta(0)$ (the initial displacement); if $\theta(0) = 0$ we say the pendulum has any period (this is common convention).

Linearization at $\theta = 0$ vs. Main ODE



Solutions $\theta(t)$ to linearization at $\theta = 0$ for $\theta(0) = \pi/6$ (thin), $\theta(0) = 2\pi/6$ (medium), and $\theta(0) = 3\pi/6$ (thick).

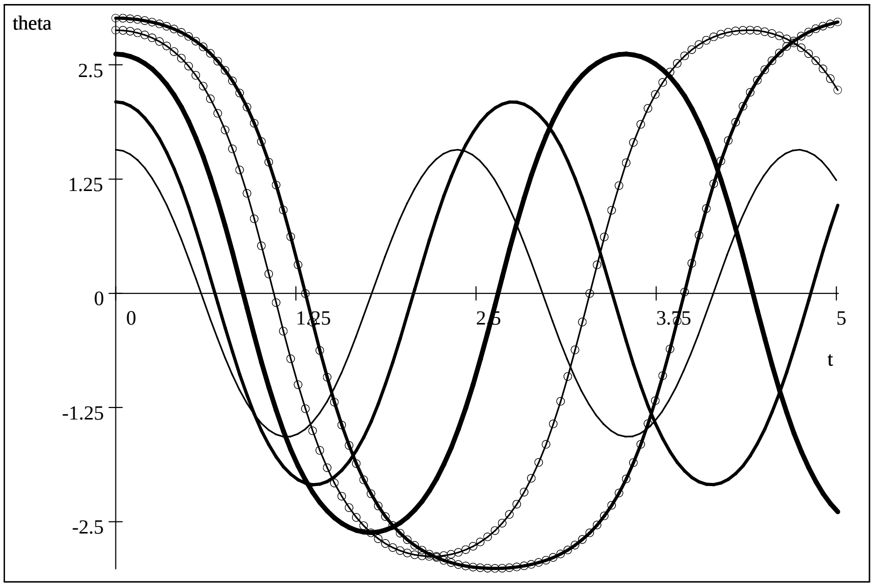


Solutions $\theta(t)$ to main ODE (1) for $\theta(0) = \pi/6$ (thin), $\theta(0) = 2\pi/6$ (medium), and $\theta(0) = 3\pi/6$ (thick).

Exercise 1 *Compare and contrast the two plots above. What are the main features to note? Recall the discussion of Galileo at the end of the previous handout.*

Bounds for Actual Period using Linearization

The second plot above shows that the actual period seems to increase as $\theta(0)$ increases in magnitude. Let us explore this further, first numerically:



Solutions $\theta(t)$ to main ODE (1) for $\theta(0) = 3\pi/6$ (thin), $\theta(0) = 4\pi/6$ (medium), $\theta(0) = 5\pi/6$ (thick), $\theta(0) = 11\pi/12$ (thindot), and $\theta(0) = 23\pi/24$ (mediumdot).

The actual period continues to increase as $\theta(0)$ increases to π.

Our goal is to obtain bounds for the actual period when $\theta(0)$ is close to π.

Instead, let us bound the time it takes for the pendulum to fall from $\pi/2 < \theta(0) < \pi$ to $\pi/2$ rad.

Lower Bound via Linearization at $\theta = \pi$

This linearization:

$$\theta'' = -\frac{g}{L}(-\theta + \pi)$$

was obtained by replacing $\sin\theta$ with $-\theta + \pi$ in our ODE (1):

$$\theta'' = -\frac{g}{L}\sin\theta.$$

Note that $|\sin\theta| \leq |-\theta + \pi|$ for ALL θ, with equality only for $\theta = \pi$. Thus, by linearizing we have INCREASED the magnitude of acceleration of our pendulum. This means we have only DECREASED the time it takes to fall to $\pi/2$ (so we obtain a lower bound on the time).

The solution to our linearized ODE is:

$$\theta(t) = [\theta(0) - \pi]\cosh\left(t\sqrt{g/L}\right) + \pi.$$

Thus, we solve $\theta(t) = \pi/2$ for t (as a function of $\theta(0)$):

$$\begin{aligned}
\pi/2 &= \theta(t) = [\theta(0) - \pi]\cosh\left(t\sqrt{g/L}\right) + \pi \\
-\pi/2 &= [\theta(0) - \pi]\cosh\left(t\sqrt{g/L}\right) \\
\frac{\pi}{2[\pi - \theta(0)]} &= \cosh\left(t\sqrt{g/L}\right) \\
t\sqrt{g/L} &= \cosh^{-1}\left(\frac{\pi}{2[\pi - \theta(0)]}\right) \\
t &= \sqrt{L/g}\cosh^{-1}\left(\frac{\pi}{2[\pi - \theta(0)]}\right).
\end{aligned}$$

Recall that:

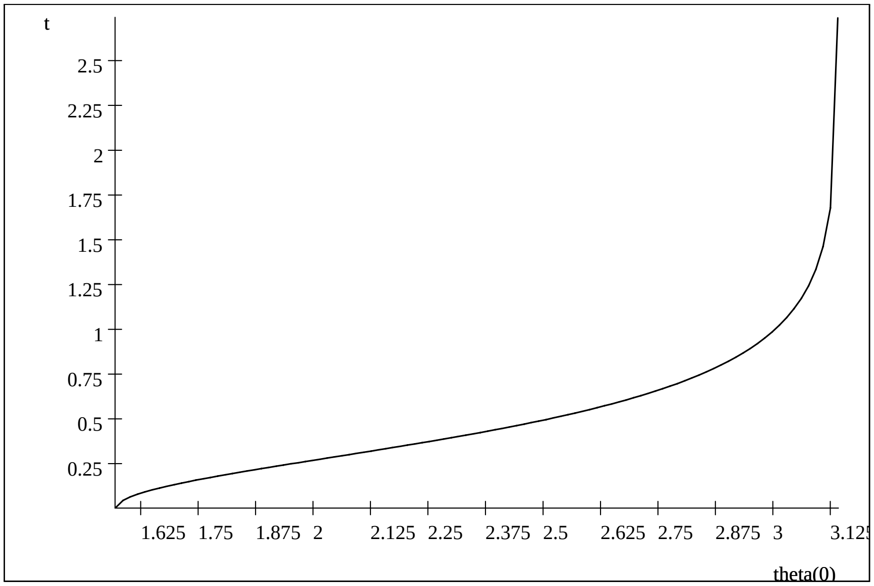
$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \text{ for } x \geq 1.$$

Exercise 2 *Verify that the use of* $\cosh^{-1}$ *above is in the valid domain.*

In this handout $L = 1$ and $g = 9.8$. So we obtain:

$$t = 1/\sqrt{9.8}\cosh^{-1}\left(\frac{\pi}{2[\pi - \theta(0)]}\right)$$

as a lower bound for the time it takes the pendulum to fall from $\theta(0)$ to $\pi/2$. We plot t as a function of $\theta(0)$:



Exercise 3 *Show that this lower bound for* t *approaches* ∞ *as* $\theta(0) \to \pi^-$.

Exercise 4 *What does the previous exercise imply?*

Let us compare this lower bound for time to fall from $\theta(0)$ to $\pi/2$ with the actual time (in s):

$\theta(0)$	Lower Bound	Actual Time
$3\pi/6$	0.000	0.000
$4\pi/6$	0.307	0.344
$5\pi/6$	0.563	0.590
$11\pi/12$	0.792	0.811
$23\pi/24$	1.015	1.033

Upper Bound

Exercise 5 *Obtain a useful upper bound for the time to fall from $\theta(0)$ to $\pi/2$.*

Hint: replace $\sin\theta$ in (1) with a linear function of θ that DECREASES the magnitude of acceleration.