**Ideal Pendulum Handout**
**Math 427K**
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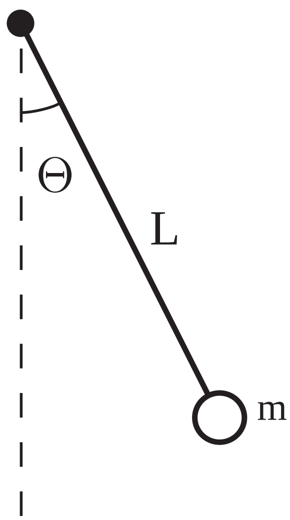


Figure 1: Ideal Pendulum

The ideal pendulum above has a mass $m$ kg attached to a rigid rod of length $L$ m. The other end of the rigid rod is hinged. We say *ideal* pendulum since we assume: the hinge operates without friction, the rigid rod is massless, there is no air resistance, the acceleration due to gravity is constant $g$ $\mathrm{m/s^2}$ and no forces other than gravity act on the mass $m$. Understanding the ideal pendulum is a logical first step, after which one can bring other forces into consideration.

By convention, we choose $g > 0$. This means on the earth $g = +9.8$ $\mathrm{m/s^2}$ and on the moon $g = +1.67$ $\mathrm{m/s^2}$. We leave the actual value unspecified but we stick to the convention that $g > 0$.

The position of the pendulum is described by the real number $\theta$ measured in radians (recall: radians are a dimensionless quantity). Hanging straight down corresponds to $\theta = 0$, pointing straight up corresponds to $\theta = \pm\pi$. An increase in $\theta$ corresponds to counterclockwise (CCW) motion, and a decrease in $\theta$ corresponds to clockwise (CW) motion. Thus, in Figure 1 above, $\theta$ is small and positive, approximately $\pi/6$ radians or 30 degrees. Of course, changing $\theta$ by $2\pi$ radians leaves the pendulum in the same position. In particular, $\theta = 2\pi k$ corresponds to hanging straight down for any integer $k$.

As derived in class, the ODE governing the motion of the ideal pendulum is:

$$\theta'' = -\frac{g}{L}\sin\theta. \tag{1}$$

Here $\theta = \theta(t)$ is a function of time $t$ and $\theta'' = d^2\theta/dt^2$. This is a second order (two derivatives) nonlinear (the presence of $\sin\theta$) ODE. Two initial conditions are required to specify a solution, namely $\theta(0)$ and $\theta'(0)$.

**Remark 1** *We will always assume that* $-\pi < \theta(0) \leq \pi$ *and* $\theta'(0) = 0$. *The first assumption is fine since any value of* $\theta(0)$ *outside of this range corresponds to a value of* $\theta(0)$ *in this range. The second assumption means physically that one moves the pendulum to* $\theta(0)$ *and releases it without pushing, only gravity acts.*

**Remark 2** *This ODE has a unique solution for any given initial conditions (a solution exists and it is unique). This result can be proved mathematically and is physically comforting (why?).*

Even though solutions exist and are unique, they cannot be found in an elementary way. The work below is devoted to understanding the behavior of solutions even though we cannot explicitly find them.

# Equilibrium Solutions

Recall that equilibrium solutions are constant solutions. Suppose $\theta(t) = c$ is a constant solution to (1). Then $\theta'(t) = 0$ and $\theta''(t) = 0$. By (1), $\sin(\theta(t)) = 0$ which means that $c = k\pi$ for some integer $k$. As $\theta(0) \in (-\pi, \pi]$, we see that $c = 0$ or $c = \pi$. That is, the equilibrium solutions to (1) are:

$$\begin{aligned} \theta_0(t) &= 0, \text{ and} \\ \theta_\pi(t) &= \pi. \end{aligned}$$

The solution $\theta_0(t) = 0$ corresponds to starting the pendulum straight down: it does not move. The solution $\theta_\pi(t) = \pi$ corresponds to starting the pendulum straight up: again it does not move (it takes a steady hand to start a pendulum in this position). There are no other equilibrium solutions.

# Stability

Recall that we only ask whether an *equilibrium* solution is stable or unstable. An equilibrium solution is *stable* provided: any other solution with initial conditions *sufficiently close to* the equilibrium solution stays close to the equilibrium solution. An equilibrium solution is *unstable* provided: some solutions have initial conditions close to the equilibrium solution but they move farther away in some future time.

We only have two equilibrium solutions to investigate. As discussed in class, it is easy to see physically that $\theta_0$ is stable while $\theta_\pi$ is unstable (why?). This can also be seen from looking at the sign of $\theta''$ for a given initial value of $\theta(0)$.

**Example 3** *Suppose* $\theta(0) = 3.1$. *That is, we start the pendulum nearly straight up (close to the equilibrium solution* $\theta_\pi$*) but just a bit to the right. Then:*

$$\theta''(0) = -\frac{g}{L}\sin(\theta(0)) < 0.$$

*Since* $\theta'(0) = 0$ *(we always assume this!), we see that the pendulum starts at* $\theta(0) = 3.1$ *and has a negative initial acceleration. This means* $\theta$ *decreases initially and so the pendulum swings CW away from the equilibrium solution.*

The reader should perform similar investigations for $\theta(0)$ near and on either side of 0 and $\pi$ and conclude that $\theta_0$ is stable and $\theta_\pi$ is unstable.

# Linearization

A useful technique for understanding nonlinear equations like (1) is to work with their linearizations. We will linearize (1) near each of its two equilibrium solutions. The reader is already familiar with linearization from calculus (i.e. the tangent line to the graph); tangent lines change from point to point, just as the linearization of an ODE depends on the initial conditions.

## Linearization at $\theta = \pi$

The nonlinear term in (1) is $\sin\theta$. Near $\theta = \pi$ we know that $\sin\theta \approx -\theta + \pi$ (why?). Thus, we simply replace $\sin\theta$ with $-\theta + \pi$ in (1) to obtain the linearization at $\theta = \pi$:

$$\begin{aligned} \theta'' &= -\frac{g}{L}(-\theta + \pi), \text{ or} \\ \theta'' - \frac{g}{L}\theta &= -\frac{g\pi}{L}. \end{aligned} \tag{2}$$

This is a linear *non*homogenous ODE of order 2 with constant coefficients. The associated homogeneous equation is:

$$\theta'' - \frac{g}{L}\theta = 0,$$

which has general solution:

$$\theta(t) = c_1 \exp\left(t\sqrt{g/L}\right) + c_2 \exp\left(-t\sqrt{g/L}\right).$$

Also, a particular solution of (2) is:

$$\theta(t) = \pi.$$

This implies that the general solution to (2) is:

$$\theta(t) = c_1 \exp\left(t\sqrt{g/L}\right) + c_2 \exp\left(-t\sqrt{g/L}\right) + \pi.$$

Now, using the initial conditions $\theta(0) =$ (value close to $\pi$) and $\theta'(0) = 0$ we get the solution to (2):

$$\theta(t) = \left(\frac{\theta(0) - \pi}{2}\right)\left(\exp\left(t\sqrt{g/L}\right) + \exp\left(-t\sqrt{g/L}\right)\right) + \pi.$$

What does this solution say? If $\theta(0) = \pi$ we get the constant solution $\theta(t) = \pi$ (as we should!). What happens if $\theta(0)$ is near but unequal to $\pi$? Convince yourself that the solution moves away from $\theta(0)$. Thus, we see $\theta_\pi$ is unstable using the linearization. How long does it take for the pendulum to swing away from $\theta(0)$? We will discuss this more later.

## Linearization at $\theta = 0$

The nonlinear term in (1) is $\sin\theta$. Near $\theta = 0$ we know that $\sin\theta \approx \theta$. Thus, we simply replace $\sin\theta$ with $\theta$ in (1) to obtain the linearization at $\theta(0) = 0$:

$$\begin{aligned} \theta'' &= -\frac{g}{L}\theta, \text{ or} \\ \theta'' + \frac{g}{L}\theta &= 0. \end{aligned} \tag{3}$$

This is a linear homogenous ODE of order 2 with constant coefficients which we can completely solve! Note that this is an equation for simple harmonic motion (a restoring force). The roots of the characteristic equation are: $r = \pm i\sqrt{g/L}$. Using our initial conditions $\theta(0) =$ (value close to 0) and $\theta'(0) = 0$, we see that the solution (3) is:

$$\theta(t) = \theta(0)\cos\left(t\sqrt{g/L}\right). \tag{4}$$

The simple harmonic motion is staring us in the face.

**Example 4** *Let $\theta(0) = \pi/6$ as in Figure 1 and let $g/L = 1$. Then we have the solution:*

$$\theta(t) = \frac{\pi}{6}\cos(t).$$

In (4), $\theta(0)$ is called the *amplitude* (why?). The *period* is the time it takes for the pendulum to make one complete oscillation, which is plainly:

$$\text{Period } = 2\pi\sqrt{L/g}.$$

It is amazing that $\theta(0)$ (the initial displacement) does not appear in the period! Think about this! Galileo was the first to make this observation (c.1602): *the period of a pendulum is independent of its initial displacement*; it only depends on the acceleration due to gravity $g$ and the length of the pendulum's arm $L$. This observation is important for clock makers (why?).

We will investigate solutions further in the next handout. Was Galileo really correct? What assumptions have we made to conclude Galileo's observation? Are they valid? Why or why not?