<u>Claim</u>: If $f: X \to Y$ is continuous and proper and Y is locally compact and Hausdorff, then f is a closed map.

pf: Let $\emptyset \neq A \subseteq X$ be closed. Let $y \in Y - f(A)$.

Suppose $\nexists\, U$ open in Y s.t. $y \in U \subseteq Y - f(A)$.

Let K be a compact nhbd of y in Y containing ~~the~~ an open nhbd V of y in Y.
(exists since Y is locally compact).

Let J be the directed set of open nhbds of y contained in V partially ordered by reverse inclusion.

For each $U \in J$, $\exists\, z_U \in U \cap f(A)$.

Define the net
$$\begin{aligned} J &\xrightarrow{\theta} Y \\ U &\longmapsto z_U. \end{aligned}$$

$\theta \to y$
pf: Let U be any open nhbd of y in Y.
Consider $U \cap V \in J$. If $W \geq U \cap V$, then $z_W \in W \subseteq U \cap V \subseteq U$ //

$f^{-1}(K)$ is compact in X since f is proper.

For each $U \in J$, $z_U \in K \cap f(A)$ and so $\exists$
$a_U \in f^{-1}(K) \cap A$ s.t. $f(a_U) = z_U$.

Define the net
$$\begin{aligned} J &\xrightarrow{\psi} X \\ U &\longmapsto a_U \end{aligned}$$
and note that $\psi: J \to f^{-1}(K) \cap A$.

As $f^{-1}(K)$ is compact, ψ has a convergent subnet. That is, $\exists$ a directed set J' and a final function $\mu: J' \to J$ s.t. the net $\psi \circ \mu : J' \to X$ converges. Say $\psi \circ \mu \to a \in A$.

$\boxed{\text{since } A \text{ is closed}}$

Now, f is continuous and so: $f \circ (\psi \circ \mu) \to f(a) \in f(A)$.

$$f \circ (\psi \circ \mu) = (f \circ \psi) \circ \mu = \theta \circ \mu \to y.$$

$\theta \circ \mu$ is a subnet of θ and $\theta \to y$, hence $\theta \circ \mu \to y$.

As Y is Hausdorff, $y = f(a) \in f(A)$, a contradiction. //

Cor: if, additionally, f is surjective, then f is a quotient map.