Borromean rays and hyperplanes

Jack S. Calcut[*,‡,**], Jules R. Metcalf-Burton[*,§], Taylor J. Richard[*,¶]
and Liam T. Solus[†,‖,‡‡]

*Department of Mathematics, Oberlin College,
Oberlin, OH 44074, USA*

*†Department of Mathematics, University of Kentucky,
Patterson Office Tower Room 722,
Lexington, KY 40504, USA*
‡jcalcut@oberlin.edu
§jmetcalf@oberlin.edu
¶trichard@oberlin.edu
‖liam.solus@uky.edu
**http://www.oberlin.edu/faculty/jcalcut/
‡‡http://www.ms.uky.edu/~solusl

ABSTRACT

Three disjoint rays in $\mathbb{R}^3$ form *Borromean rays* provided their union is knotted, but the union of any two components is unknotted. We construct infinitely many Borromean rays, uncountably many of which are pairwise inequivalent. We obtain uncountably many Borromean hyperplanes.

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1. Introduction

For proper, locally flat embeddings in $\mathbb{R}^n$, it is well known that:

$$A \text{ ray } (= \text{ copy of } [0, \infty)) \text{ knots if and only if } n = 3. \tag{1.1}$$

$$A \text{ hyperplane } (= \text{ copy of } \mathbb{R}^{n-1}) \text{ knots if and only if } n = 3. \tag{1.2}$$

Both facts hold in the smooth, piecewise linear, and topological categories [3]. Fox and Artin discovered the first knotted ray [10]. The boundary of a closed regular neighborhood of any knotted ray is a knotted hyperplane. For $n > 3$, fact (1.2) is the Cantrell–Stallings hyperplane unknotting theorem, an enhancement of the famous Schoenflies theorem of Mazur and Brown [3; 25, p. 98]. Embeddings in $\mathbb{R}^n$, $n \neq 3$, of at most countably many rays or hyperplanes were recently classified by

King, Siebenmann, and the first author [3]. In $\mathbb{R}^3$, no classification is known or even conjectured.

A ray or multiray $r \subset \mathbb{R}^3$ is *unknotted* if and only if an automorphism of $\mathbb{R}^3$ carries r to a disjoint union of radial rays. Unknotted multirays with the same number of components are ambient isotopic [3, Lemma 4.1].

A rich collection of knotted rays may be obtained from wild arcs. Let $a \subset S^3$ be an arc with one wild (= non-locally flat) point p. Consider $r = a - \{p\}$ in $\mathbb{R}^3 = S^3 - \{p\}$. If p is an endpoint of a, then r is a knotted ray. If p is an interior point of a, then r is a knotted, two-component multiray. Hence, in $\mathbb{R}^3$:

$$\text{There exist infinitely many knot types of a ray [1].} \tag{1.3}$$

$$\text{There exist uncountably many knot types of a ray [18].} \tag{1.4}$$

$$\text{There exist uncountably many knot types of two-component}$$
$$\text{multirays with unknotted components [11].} \tag{1.5}$$

A three-component multiray $r \subset \mathbb{R}^3$ will be called *Borromean rays* provided r is knotted, but any two components of r form an unknotted multiray. Debrunner and Fox constructed an example equivalent to Borromean rays [5]. Earlier, Doyle attempted a construction [6], but his argument contained a gap [5]. We prove that there exist uncountably many knot types of Borromean rays. The following is an overview.

Consider the four blocks in Fig. 1. The block A consists of a three-component tangle T in a thickened 2-sphere $S^2[1, 2]$. Any two components of T can be straightened by an ambient isotopy of $S^2[1, 2]$ relative to boundary. However, no diffeomorphism of $S^2[1, 2]$ sends T to a radial tangle (Corollary 2.6). The blocks $\overline{A}$, A^*, and $\overline{A}^*$ are reflections of A. Let $\mathcal{B} = \{A, \overline{A}, A^*, \overline{A}^*\}$ be the set of these four blocks. Let B_i, $i \in \mathbb{Z}^+$, be a sequence of blocks in $\mathcal{B}$. The *infinite concatenation* $D^3 B_1 B_2 B_3 \cdots$ is obtained by gluing the inner boundary sphere of B_1 to the boundary of a 3-disk, and gluing the inner boundary sphere of B_{i+1} to the outer boundary sphere of B_i for each $i \in \mathbb{Z}^+$. This yields the pair:

$$(\mathbb{R}^3, \tau) = D^3 B_1 B_2 B_3 \cdots, \tag{1.6}$$

where $\tau \subset \mathbb{R}^3$ is a three-component multiray. Each such τ forms Borromean rays (Corollary 4.9). Let σ be the Borromean rays determined by another such sequence C_i, $i \in \mathbb{Z}^+$. We prove that if $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$ is a diffeomorphism of pairs, then there is an isotopy of f to a diffeomorphism $g : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$ and an integer n such that:

$$g(B_i) = C_{i+n} \quad \text{for all sufficiently large } i \in \mathbb{Z}^+. \tag{1.7}$$

Hence, the existence of f boils down to: (i) the tails of the sequences B_i and C_i, and (ii) possible diffeomorphisms between individual blocks in $\mathcal{B}$. The latter are studied in Sec. 3. Our main result, Theorem 8.1, gives necessary and sufficient conditions for two such sequences to yield equivalent Borromean rays. Care is taken to account
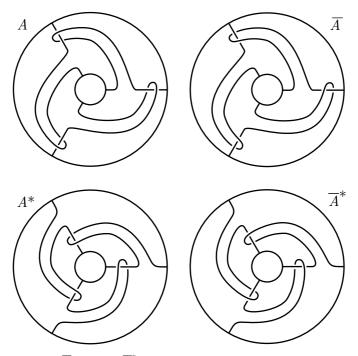
Fig. 1.   Four blocks A, $\overline{A}$, A^*, and $\overline{A}^*$. Each block is a three-component tangle in a thickened 2-sphere. The set of these four blocks is denoted by $\mathcal{B}$.

for orientation. As an application, we give necessary and sufficient conditions for our Borromean rays to be achiral (Corollary 8.3). While most turn out to be chiral, we give a countably infinite family of pairwise inequivalent, achiral Borromean rays.

The notion of an *irreducible block* plays a central role. A block B is *irreducible* provided: if B is diffeomorphic to a concatenation $B_1 B_2$, then B_1 or B_2 is diffeomorphic to a *trivial block* (= block with a radial tangle). Trivial blocks are irreducible (Proposition 4.5). We use this fact to prove that each τ in (1.6) forms Borromean rays. The block A is also irreducible (Theorem 6.1), although the proof is more technical. Thus, blocks in $\mathcal{B}$ are irreducible. This fact is used to improve diffeomorphisms as in (1.7).

We discovered the block A as follows. Consider a three-component multiray $\tau \subset \mathbb{R}^3$ with the property:

(†) Any two components of τ form an unknotted multiray.

Let τ_i, $i \in \{1, 2, 3\}$, denote the components of τ. Property (†) implies that for each pair τ_i and τ_j of components of τ there is a *strip* $S_{i,j} \subset \mathbb{R}^3$ (= properly embedded copy of $[0, 1] \times [0, \infty)$) whose *stringers* (= $\{0\} \times [0, \infty)$ and $\{1\} \times [0, \infty)$) equal τ_i and τ_j. The interior of $S_{i,j}$ probably intersects the third component of τ (if not, then τ is unknotted). Using a small regular neighborhood of τ, one may twist these
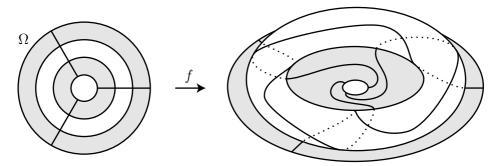
Fig. 2. Immersion f from the compact, two-dimensional annulus Ω into the thickened sphere $S^2[1,2]$. The image of f is a torus sitting atop an annulus and containing the tangle T.

strips about their stringers so that they patch together to form a general position immersion $f : S^1 \times [0, \infty) \looparrowright \mathbb{R}^3$. Let p_i, $i \in \{1, 2, 3\}$, be equally spaced points in S^1. The immersion f sends the radial ray $\{p_i\} \times [0, \infty)$ to τ_i for each $i \in \{1, 2, 3\}$ and is an embedding on each of the three closed sectors between such radial rays. In an attempt to unknot τ, one may try to eliminate multiple points of f. Certain types of multiple points can be eliminated. However, difficulties arise from essential circles of double points. Figure 2 displays the relevant, compact part of a simple configuration where two essential circles of double points are identified under f. Here, the domain of f is the compact, two-dimensional annulus Ω containing three radial arcs. The image of these three radial arcs is the tangle T in the block A. Having found A, a fundamental group calculation shows that A is not a trivial block (see Sec. 2.2). Our proof that A is irreducible (see Sec. 6) makes essential use of the immersion f in Fig. 2.

After we discovered the block A, we found Debrunner and Fox's *mildly wild* 3-*frame* [5; 25, pp. 95–97] and Doyle's attempted example [6]. We pause to make some observations on these two examples.

(1) Debrunner and Fox's mildly wild 3-frame $D \subset \mathbb{R}^3$ is compact, periodic, and contains one wild point o. In our notation, their building block is the concatenation $\overline{A} \, \overline{A}^*$. Put D in S^3. We define the *Debrunner–Fox Borromean rays* to be $\delta = D - \{o\}$ in $\mathbb{R}^3 = S^3 - \{o\}$. In our notation:

$$(\mathbb{R}^3, \delta) = D^3 AA^* AA^* AA^* \cdots.$$

By Corollary 8.3 below, δ is achiral.

(2) Debrunner and Fox's proof that D is wild hinges on showing a certain group is not finitely generated. Their approach yields a mildly wild n-frame for each $n \geq 3$. On the other hand, it is not clear how one can use it to distinguish between two wild 3-frames. In Sec. 4, we use irreducibility of trivial blocks to prove our multirays (including δ) are knotted. Then, we use irreducibility of blocks in $\mathcal{B}$ to distinguish between multirays.

(3) Bing showed that Doyle's 3-frame is standard [5], though Bing's argument is not indicated. Lemma 2.13 below is useful for recognizing unknotted multirays and applies to Doyle's 3-frame.

We are indebted to Scott Taylor for pointing out to us some relations between the present work and the theory of knotted theta graphs. By crushing the inner and outer boundary spheres of our block A each to a point, one obtains Kinoshita's theta curve [15, 26, 17]. The complement of an open regular neighborhood of T in A is hyperbolic by Thurston [29, p. 135].

This paper is organized as follows. Section 2 presents conventions and notation, introduces blocks (including several examples), and proves some basic properties concerning blocks. Section 3 studies diffeomorphisms between individual blocks in $\mathcal{B}$. Section 4 introduces irreducible blocks, proves trivial blocks are irreducible, deduces some corollaries, and constructs infinitely many irreducible blocks containing two-component tangles. Section 5 identifies some unknotted ball-arc pairs in blocks. Section 6 proves that blocks in $\mathcal{B}$ are irreducible. Section 7 simplifies certain spheres in concatenations of Borromean blocks and deduces two useful corollaries. Section 8 classifies Borromean rays arising from sequences of blocks in $\mathcal{B}$ and then uses regular neighborhoods of multirays to obtain results on knotted multiple hyperplane embeddings. In particular, we prove that there exist uncountably many pairwise inequivalent so-called *Borromean hyperplanes* in $\mathbb{R}^3$.

2. Building Blocks

We work in the smooth ($= C^\infty$) category. Throughout, $\approx$ denotes diffeomorphism of manifolds or manifold pairs. A map is *proper* provided the inverse image of each compact set is compact. All isotopies will be smooth and proper. A submanifold $X \subset M$ is *neat* provided $\partial X = X \cap \partial M$ and this intersection is transverse [14, pp. 30–31; 16, pp. 27, 31, 62].

A *ray* is a proper embedding of $[0, \infty)$. A *multiray* is a proper embedding of $Z \times [0, \infty)$ where Z is a finite or countably infinite discrete space. Indeed, each embedded submanifold of $\mathbb{R}^n$ contains at most countably many components since $\mathbb{R}^n$ is a separable metric space. A ray in $\mathbb{R}^n$ is *radial* provided either it is straight and emanates from the origin, or it is contained in such a ray. In particular, a radial ray can meet the origin only at its endpoint. A collection of intervals embedded in $\mathbb{R}^n$ is *radial* provided each component lies in a radial ray.

The standard euclidean norm on $\mathbb{R}^n$ is $\|x\| := (\sum x_i^2)^{1/2}$. On $\mathbb{R}^3$, the euclidean norm function will be denoted:

$$
\begin{aligned}
\mathbb{R}^3 &\xrightarrow{\ \eta\ } \mathbb{R} \\
x &\longmapsto \|x\|.
\end{aligned}
\tag{2.1}
$$

All lengths come from the standard euclidean metric. The unit n-disk D^n consists of all points $p \in \mathbb{R}^n$ such that $\|p\| \leq 1$. The unit $(n-1)$-sphere is $S^{n-1} = \partial D^n$. The

sphere of radius $t > 0$ in $\mathbb{R}^n$ about 0 is denoted $S^{n-1}[t]$ and is called a *level sphere*. In particular, $S^2 = S^2[1]$. Let $S^{n-1}[t_1, t_2]$, where $0 < t_1 < t_2$, denote the thickened sphere of points $p \in \mathbb{R}^n$ such that $t_1 \leq \|p\| \leq t_2$. In particular, $\partial S^{n-1}[t_1, t_2]$ equals the disjoint union of the spheres $S^{n-1}[t_1]$ and $S^{n-1}[t_2]$. Let $S^{n-1}[t, \infty)$ denote the half-infinite annulus of points $p \in \mathbb{R}^n$ such that $\|p\| \geq t$.

2.1. *Blocks*

Let $S^2[t_1, t_2]$, where $0 < t_1 < t_2$, be a thickened sphere in $\mathbb{R}^3$. A *tangle* τ is an embedding of the disjoint union of $n \geq 1$ copies of $[0, 1]$ as a neat submanifold of $S^2[t_1, t_2]$. If τ_k is a component of τ, then the initial point p_k of τ_k must lie in $S^2[t_1]$ and the terminal point must equal $(t_2/t_1)p_k \in S^2[t_2]$. So, τ_k stretches between the two boundary 2-spheres of $S^2[t_1, t_2]$, and its initial and terminal points lie on a radial ray.

A *block* is a pair $(S^2[t_1, t_2], \tau)$ where τ is a tangle. Each block is oriented: $S^2[t_1, t_2]$ inherits its orientation from the standard one on $\mathbb{R}^3$, and each component of τ is oriented to point out from the inner boundary 2-sphere. A *diffeomorphism* of blocks is any diffeomorphism of the corresponding pairs of spaces, not necessarily orientation or boundary preserving in any sense.

A *trivial block* is any pair ε_n consisting of an n-component, radial tangle in a thickened sphere (see Fig. 3).

By our convention, in every displayed block, the positive x-axis points horizontally to the right, the positive y-axis points vertically up, and the positive z-axis points out of the page towards the reader.

Given any block $B = (S^2[t_1, t_2], \tau)$, define two blocks:

$$\overline{B} := (S^2[t_1, t_2], \overline{\tau}) \text{ is the reflection of } B \text{ across the } xy\text{-plane.} \qquad (2.2)$$

$$B^* := (S^2[t_1, t_2], \tau^*) \text{ is the inversion of } B \text{ across } S^2[(t_1 + t_2)/2]. \qquad (2.3)$$

If $B = (S^2[1, 2], \tau)$, then inversion is $p \mapsto (3 - \|p\|)\frac{p}{\|p\|}$. Components of B^* are still oriented out from the inner boundary 2-sphere. Evidently, the bar and star operations commute and are involutions:

$$\overline{B}^* = \overline{B^*}, \quad \overline{\overline{B}} = B, \quad \text{and} \quad (B^*)^* = B.$$
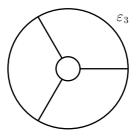

Fig. 3. Trivial block ε_3 containing a three-component, radial tangle.

Figure 1 above introduced four blocks important for our purposes. Let

$$\mathcal{B} = \{A, \overline{A}, A^*, \overline{A}^*\}$$

be the set of these four blocks. Note that $\mathcal{B}$ is closed under the bar and star operations. By construction, blocks in $\mathcal{B}$ are pairwise diffeomorphic. They are pairwise distinct, though, up to finer equivalence relations, as explained below.

2.2. *Equivalence relations on blocks*

The coarsest equivalence relation on blocks we consider is that of *diffeomorphism* (defined above). Finer diffeomorphism relations, involving orientation and/or boundary preservation, arise in Sec. 3. On blocks with the same underlying thickened spheres, the finest relation we consider (short of equality) is that of *ambient isotopy relative to boundary*, meaning ambient isotopy of tangles fixing both boundary 2-spheres pointwise at all times.

Example 2.1 (One-Component Tangles). Any block $B = (S^2[1, 2], \tau)$, where τ has one component, is ambient isotopic relative to boundary to ε_1. This fact is well-known (e.g. it is an exercise in Rolfsen [24, p. 257]). We are not aware of a published proof, so we sketch one. All isotopies are ambient and relative to boundary. By a preliminary isotopy (left to the reader), we assume B appears as in Fig. 4 where K is a crossing diagram in general position. It suffices to prove that crossings of K may be switched by isotopy, since then we may arrange that K has monotonic z-coordinate and the result follows. So, consider a crossing C of K with over arc α and under arc β. Push α and β sufficiently close together in the z-direction. Let p and q be the midpoints of α and β respectively, where p lies directly above q. Let $[0, t]$ denote the subarc of τ where $0 \in S^2$ and $t \in \text{Int } \tau$. Let $J = [0, p]$ if $q \notin [0, p]$, and let $J = [0, q]$ if $q \in [0, p]$. In other words, J is the unique subarc of τ originating on S^2 and terminating at the first point, p or q, encountered by J. Assume $J = [0, p]$ (otherwise, flip the picture over). Isotop q close to S^2 by following just underneath J and stretching β. Then, loop β under S^2 and isotop q back (again using J as a guide) to lie above p. The crossing C has been switched, completing the proof.
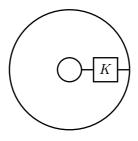

Fig. 4.   Block B with a one-component tangle τ.

Fig. 5.   Ray $r \subset \mathbb{R}^3$ obtained by tying successive knots in a radial ray.

Remark 2.2. Example 2.1 has the following possibly surprising corollary, which appears to be due to Wilder [10, p. 987]. If the ray $r \subset \mathbb{R}^3$ is obtained by tying successive knots in a radial ray (see Fig. 5), then r is ambient isotopic to a radial ray.

Proof. Let $K_i \subset S^2[i, i+1]$ for each $i \in \mathbb{Z}^+$. Simultaneously apply the straightening process from Example 2.1 to each $S^2[i, i+1]$. □

Example 2.3 (Two-Component Tangles). Let $k \subset S^3$ be a knot. Let $D \subset S^3$ be a 3-disk such that: (i) $a := k \cap D$ is a neatly embedded arc in D, and (ii) (D, a) is an unknotted ball-arc pair. Let $D' \subset \text{Int } \varepsilon_2$ be a small, round 3-disk meeting one tangle component in an arc. The *knot block* $B(k)$ is obtained from ε_2 by replacing D' with $S^3 - \text{Int } D$ as in Fig. 6. In general, $B(k)$ is well-defined up to diffeomorphism. If k itself is oriented, then one could define $B(k)$ more carefully. Let τ be the tangle in a knot block $B(k)$. Evidently, deleting the boundary from $B(k) - \tau$ yields $S^3 - k$. In particular, $B(k) \approx B(k')$ implies $S^3 - k \approx S^3 - k'$. So, knots with nonisomorphic groups (e.g. torus knots [2, p. 47]) yield nondiffeomorphic knot blocks. Finally, let $B = (S^2[1, 2], \tau)$ be a block where τ has two components, τ_1 and τ_2. Then, B is ambient isotopic relative to boundary to some knot block. To see this, straighten τ_2 using the process in Example 2.1. Then push τ_1 away from τ_2 by integrating a suitable vector field tangent to level spheres.

Example 2.4 (Dirac's Block). Consider the block D in Fig. 7. The thickened sphere underlying D is $S^2[1, 2]$, and D is obtained from ε_3 by fixing the inner boundary 2-sphere pointwise and rigidly rotating the outer boundary sphere one revolution about the x-axis. In particular, $\varepsilon_3 \approx D$, in fact by an orientation preserving diffeomorphism that is pointwise the identity on both boundary 2-spheres. On the other hand, ε_3 and D are not ambient isotopic relative to boundary, as
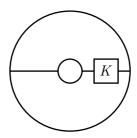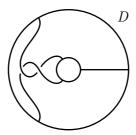

Fig. 6.   Knot block $B(k)$ where K is a diagram yielding k.

Fig. 7.   Dirac's block D.

proved by Newman [23] and Fadell [7] (see also [8; 22, Sec. 11.1–11.2]). If D had been obtained from ε_3 by two complete twists, rather than just one, then D would have been ambient isotopic relative to boundary to ε_3 by Dirac's belt trick.

A diffeomorphism between thickened spheres is: (i) *radial* provided it sends radial arcs to radial arcs, and (ii) *level* provided it sends level 2-spheres to level 2-spheres. The next lemma says that there is essentially just one trivial block ε_n for each $n \in \mathbb{Z}^+$.

Lemma 2.5 (Trivial Block Uniqueness). *Let $\varepsilon_n = (S^2[t_1, t_2], r)$ and $\varepsilon'_n = (S^2[t'_1, t'_2], r')$ be trivial blocks. Then, there is a radial, level, orientation preserving diffeomorphism $h : \varepsilon_n \to \varepsilon'_n$ sending $S^2[t_i]$ to $S^2[t'_i]$ for $i = 1, 2$. If $[t_1, t_2] = [t'_1, t'_2]$, then there is an ambient isotopy H_t, $0 \leq t \leq 1$, of $S^2[t_1, t_2]$ such that:*

$$H_0 = \mathrm{Id} \ and \ H_1 = h. \tag{2.4}$$

$$H_t \ is \ a \ radial, \ level \ diffeomorphism \ for \ all \ 0 \leq t \leq 1. \tag{2.5}$$

Proof. Let $\psi : [t_1, t_2] \to [t'_1, t'_2]$ be the unique affine, orientation preserving diffeomorphism. Then, $x \mapsto \psi(\|x\|) \cdot x/\|x\|$ is a radial, level diffeomorphism $S^2[t_1, t_2] \to S^2[t'_1, t'_2]$. So, it suffices to consider the case $[t_1, t_2] = [t'_1, t'_2] = [1, 2]$. Let $p_i \in S^2$, $1 \leq i \leq n$, denote the initial points of the components r_i of r. Define p'_i similarly for r'. Let α_1 be a smooth, simple path from p_1 to p'_1 in S^2. Let $\nu\alpha_1$ be a smooth regular neighborhood of α_1 in S^2. There is an ambient isotopy of S^2, with support in $\nu\alpha_1$, carrying p_1 to p'_1. For instance, begin with a suitable nonzero tangent vector field to α_1, extend to a vector field v on S^2 that vanishes outside of $\nu\alpha_1$, and then integrate v (cf. [19, pp. 22–24]). Extend this isotopy radially to get an ambient isotopy of $S^2[1, 2]$ carrying r_1 to r'_1. Any component r_i, $i \geq 2$, that moved during this isotopy is still radial and is still denoted r_i. Repeat this procedure, while choosing $\nu\alpha_i$ disjoint from $p'_1, \ldots, p'_{i-1}$. $\qquad \square$

To distinguish A from ε_3 up to diffeomorphism, it suffices to distinguish the fundamental groups of their tangle complements up to isomorphism. Presentations of such groups are obtained using Wirtinger's algorithm. Consider the diagram
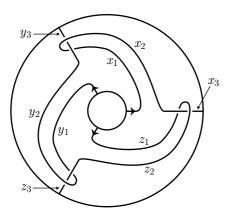
Fig. 8.   Block A with oriented and labeled arcs.

of A in Fig. 8. As usual, labels of arcs correspond to generators of $\pi_1(A - T)$.
The basepoint is above the page. The based loop representing a generator g first
penetrates the plane of the page at a point just to the right of the oriented arc
labeled g, and has linking number $+1$ with this oriented arc. A presentation of
$\pi_1(A - T)$ is:

$$\left\langle \begin{array}{l} x_1, x_2, x_3, \\ y_1, y_2, y_3, \\ z_1, z_2, z_3 \end{array} \middle| \begin{array}{c} y_2 x_2 = x_1 y_2, x_2 y_3 = y_2 x_2, x_2 z_2 = z_1 x_2, \\ z_2 x_3 = x_2 z_2, z_2 y_2 = y_1 z_2, y_2 z_3 = z_2 y_2, \\ x_1 y_1 z_1 = 1 \end{array} \right\rangle.$$

Each crossing of T contributes a relation. The last relation is evident topologically.
It is a *vertex relation* for the fundamental group of the complement of the graph
obtained by crushing the inner boundary 2-sphere to a point [30, Sec. 4; 10, p. 981;
28, p. 148]. The outer vertex relation, $x_3 y_3 z_3 = 1$, is redundant.

For the trivial tangle, $\pi_1(\varepsilon_3 - r) \cong F_2$ is free of rank 2. To distinguish $\pi_1(A - T)$
from F_2, we count their *classes* of homomorphisms into small symmetric groups
$\mathrm{Sym}(n)$ using the computer algebra system MAGMA (a finite problem). Two homo-
morphisms $h_1, h_2 : G \to \mathrm{Sym}(n)$ are considered *equivalent* provided there exists
$\pi \in \mathrm{Sym}(n)$ such that $h_2(g) = \pi^{-1} h_1(g) \pi$ for all $g \in G$. Table 1 collects this data.
The two abelian symmetric groups are included for completeness, and the last two
rows are included for comparison with Table 4 ahead. Any of the last three rows in
Table 1 implies the following.

Corollary 2.6. *$A \not\approx \varepsilon_3$. As blocks in $\mathcal{B}$ are pairwise diffeomorphic, no block in $\mathcal{B}$
is diffeomorphic to ε_3.*

2.3. *Borromean blocks*

A *Borromean block* is a block $B = (S^2[t_1, t_2], \tau)$ such that: (i) $B \not\approx \varepsilon_3$, and (ii)
each block obtained from B by forgetting one component of τ is ambient isotopic

Table 1. Numbers of classes of homomorphisms into Sym(n).

	$\pi_1(\varepsilon_3 - r)$	$\pi_1(A - T)$
Sym(1)	1	1
Sym(2)	4	4
Sym(3)	11	11
Sym(4)	43	47
Sym(5)	161	193
Sym(6)	901	1317

relative to boundary to ε_2. By Fig. 1, each block in $\mathcal{B}$ clearly satisfies condition (ii). Corollary 2.6 implies the following.

Corollary 2.7. *The blocks A, $\overline{A}$, A^*, and $\overline{A}^*$ are Borromean.*

It is useful to observe that any block whose tangle components always "head out" is diffeomorphically trivial. Let M be a smooth closed manifold. A *tangle* τ in $M \times [0,1]$ is a neat embedding of a disjoint union of copies of $[0,1]$. In particular, $\partial\tau \subset M \times \{0,1\}$ and τ has finitely many components. In analogy with our definition of *tangle in a thickened sphere*, we might also require each component of τ to have one boundary point in $M \times \{0\}$ and one in $M \times \{1\}$. However, this property will automatically be satisfied in the following, our only use of tangles in general $M \times [0,1]$.

Lemma 2.8 (Tangle Straightening). *Let M be a smooth closed manifold. Let*

$$p : M \times [0,1] \to [0,1]$$

be projection. Let $\tau \subset M \times [0,1]$ be a tangle such that $p\,|\,\tau$ has no critical points. Then, there is an ambient isotopy H_t, $0 \leq t \leq 1$, of $M \times [0,1]$ such that:

$$H_0 = \mathrm{Id}. \tag{2.6}$$

$$H_t \,|\, M \times \{0\} = \mathrm{Id} \text{ for all } 0 \leq t \leq 1. \tag{2.7}$$

$$pH_t = p \text{ for all } 0 \leq t \leq 1. \tag{2.8}$$

$$H_1(\tau) \text{ is a finite disjoint union of straight arcs } \{x_i\} \times [0,1], \, x_i \in M. \tag{2.9}$$

Proof. Let u be a nonzero tangent vector field on τ. Since $p\,|\,\tau$ has no critical points, we may assume $u(p) > 0$ on τ. Extend u to a small tubular neighborhood U of τ so that $u(p) > 0$ on U. Then, $w := u/u(p)$ is tangent to τ, and $w(p) = 1$ on U. Using U, we get a smooth function $g : M \times [0,1] \to [0,1]$ with support in U and equal to 1 on τ. Consider the vector field $v := gw + (1-g)\,(0,1)$ on $M \times [0,1]$, where $(0,1)$ is the obvious constant vector field on $M \times [0,1]$. Let $\phi((x,s),t)$ be the maximal flow generated by v and let $\mathcal{D}$ be the domain of ϕ. Then, $p\phi((x,s),t) = s+t$ on $\mathcal{D}$ since $v(p) = 1$ on $M \times [0,1]$. As M has no boundary and $M \times [0,1]$ is compact,

patching together local flows yields:

$$\mathcal{D} = M \times \{(s, t) \mid 0 \leq s \leq 1, -s \leq t \leq 1 - s\}.$$

Let $q : M \times [0, 1] \to M$ be projection. Define the ambient isotopy H_t, $0 \leq t \leq 1$, of $M \times [0, 1]$ by:

$$H_t(x, s) := (q\phi((x, s), -st), s).$$

That is, at time t, (x, s) flows via ϕ from $M \times \{s\}$ back into $M \times \{s(1-t)\}$ and then is translated into $M \times \{s\}$. Properties (2.6)–(2.8) evidently hold, and tangency of v to τ guarantees (2.9). □

Remark 2.9. Our proof of Lemma 2.8 is a modification of the proof of the isotopy extension theorem (cf. [3, Lemma 9.15]).

Corollary 2.10 (Tangle Radialization). *Let $B = (S^2[t_1, t_2], \tau)$ be a block where τ has $n \geq 1$ components. Assume $\eta \mid \tau$ has no critical points. Then, there is an ambient isotopy G_t, $0 \leq t \leq 1$, of $S^2[t_1, t_2]$ such that*:

$$G_0 = \mathrm{Id}. \tag{2.10}$$

$$G_t \mid S^2[t_1] = \mathrm{Id} \ \textit{for all } 0 \leq t \leq 1. \tag{2.11}$$

$$\eta G_t = \eta \ \textit{on } S^2[t_1, t_2] \ \textit{for all } 0 \leq t \leq 1. \ \textit{In particular, each } G_t$$

$$\textit{is a level diffeomorphism.} \tag{2.12}$$

$$G_1(\tau) \ \textit{is radial. In particular, } G_1(B) \ \textit{equals a trivial block } \varepsilon_n. \tag{2.13}$$

Proof. Let $l : [t_1, t_2] \to [0, 1]$ be the unique affine, orientation preserving diffeomorphism. Let $h : S^2[t_1, t_2] \to S^2 \times [0, 1]$ be the diffeomorphism given by $h(x) := (x/\|x\|, l(\|x\|))$. Apply Lemma 2.8 to the tangle $h(\tau)$ in $S^2 \times [0, 1]$ and let H be the resulting isotopy of $S^2 \times [0, 1]$. The desired isotopy is $G := h^{-1} \circ H \circ (h \times \mathrm{Id})$. □

Remark 2.11. Consider a tangle τ in a Borromean block. Critical points of $\eta \mid \tau$ come in pairs since, by definition, tangle components originate on the inner boundary sphere and terminate on the outer boundary sphere. So, Corollary 2.10 implies $\eta \mid \tau$ has at least two critical points. This minimum number is achievable. A simple ambient isotopy relative to boundary of A yields the (necessarily Borromean) block A' in Fig. 9.

The next corollary will be used when concatenating blocks.

Corollary 2.12 (Tangle Radialization Near Boundary). *Consider a block $B = (S^2[t_1, t_2], \tau)$ where τ has $n \geq 1$ components. There exists $\epsilon_0 > 0$ such that,*
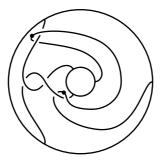
Fig. 9. Borromean block $A' = (S^2[1,2], T')$ ambient isotopic relative to boundary to A. The restriction $\eta \,|\, T'$ has just the two indicated critical points.

for each $0 < \epsilon \leq \epsilon_0$, there is an ambient isotopy K_t, $0 \leq t \leq 1$, of $S^2[t_1, t_2]$ such that:

$$K_0 = \mathrm{Id}. \tag{2.14}$$

$$K_t = \mathrm{Id} \ on \ \partial S^2[t_1, t_2] \ and \ on \ S^2[t_1 + \epsilon, t_2 - \epsilon] \ for \ all \ 0 \leq t \leq 1. \tag{2.15}$$

$$\eta K_t = \eta \ on \ S^2[t_1, t_2] \ for \ all \ 0 \leq t \leq 1. \ In \ particular, \ each \ K_t$$

$$is \ a \ level \ diffeomorphism. \tag{2.16}$$

$$K_1(\tau) \ is \ radial \ in \ S^2[t_1, t_1 + \epsilon/2] \ and \ in \ S^2[t_2 - \epsilon/2, t_2]. \tag{2.17}$$

Proof. By definition of a block, τ is neatly embedded. So, there is $\epsilon_0 > 0$ such that $\epsilon_0 < (t_1 + t_2)/2 - t_1$ and $\eta \,|\, \tau$ has no critical points in $S^2[t_1, t_1 + \epsilon_0] \cup S^2[t_2 - \epsilon_0, t_2]$. Let $0 < \epsilon \leq \epsilon_0$. Recall the diffeomorphism $h : S^2[t_1, t_2] \to S^2 \times [0, 1]$ from the proof of Corollary 2.10. Consider the tangle $h(\tau) \subset S^2 \times [0, 1]$. Define $\epsilon' := \epsilon/(t_2 - t_1)$ and note that $0 < \epsilon' < 1/2$. Construct a smooth function $b : [0, 1] \to [-\epsilon', \epsilon']$ such that:

$$b(s) = \begin{cases} s & \text{if } 0 \leq s \leq \epsilon'/2, \\ 0 & \text{if } \epsilon' \leq s \leq 1 - \epsilon', \\ s - 1 & \text{if } 1 - \epsilon'/2 \leq s \leq 1. \end{cases}$$

Also, to ensure well-definition of the isotopy to come, arrange that:

$$b(s) \in \begin{cases} [0, s] & \text{if } \epsilon'/2 \leq s \leq \epsilon', \\ [s - 1, 0] & \text{if } 1 - \epsilon' \leq s \leq 1 - \epsilon'/2. \end{cases}$$

Return now to the proof of Lemma 2.8 and redefine:

$$H_t(x, s) := (q\phi((x, s), -b(s)t), s).$$

This H is an ambient isotopy of $S^2 \times [0, 1]$. Then, $K := h^{-1} \circ H \circ (h \times \mathrm{Id})$ is the desired isotopy. $\qquad \square$

The following lemma can be useful for recognizing unknotted multirays.

Lemma 2.13 (Multiray Straightening). *Let r be a multiray in $\mathbb{R}^3$ with at most countably many components. By a small perturbation of r, assume $\eta \,|\, r$ is a Morse function. Suppose $\eta \,|\, r$ has no critical points (i.e. the rays always "head out"). Then, r is ambient isotopic to a radial multiray.*

Proof. The proof of Lemma 2.8 adapts readily (cf. [3, Lemma 9.15]). □

Remark 2.14. By Remarks 2.16(1) below, Lemma 2.13 holds provided $\eta \,|\, r_i$ has only finitely many critical points for each component r_i of r.

2.4. *Concatenations of blocks*

We define finite and countably infinite *concatenations* of blocks. In any concatenation of blocks, two natural properties are always assumed:

Block summands contain tangles with the same number, $n \geq 1$,
  of components. (2.18)

Initial points of tangles in different summands differ by a
  radial dilation. (2.19)

First, consider a finite concatenation $B_1 B_2 \cdots B_k$ of blocks B_i. By definition, this means the block $B := (S^2[1, k+1], \tau)$ where B_i has been stretched radially to coincide with $S^2[i, i+1]$. The resulting tangle τ in B is the union of the k stretched tangles. Evidently, this definition is deficient in that τ may have corners where blocks meet. Our remedy is to refine the definition: each block B_i is first adjusted, by an ambient isotopy relative to boundary provided by Corollary 2.12, so that its tangle is radial near the boundary spheres of B_i. The concatenation $B_1 B_2 \cdots B_k$ is now a block, well-defined up to (obvious) ambient isotopy relative to boundary.

The proof of the next lemma is straightforward.

Lemma 2.15. *If B is a block, then $\varepsilon_n B \approx B \approx B \varepsilon_n$.*

Each infinite concatenation $B_1 B_2 B_3 \cdots$ is defined analogously. The result is $(S^2[1, \infty), \tau)$ where τ is now a neatly embedded n-component multiray. Further, $D^3 B_1 B_2 B_3 \cdots$ denotes the pair $(\mathbb{R}^3, \tau)$ where τ is the obvious n-component multiray in $\mathbb{R}^3$. A *diffeomorphism* between infinite concatenations is a diffeomorphism between the corresponding pairs of spaces. An *ambient isotopy* between infinite concatenations is an ambient isotopy of the underlying total space carrying one multiray to the other. An ambient isotopy of $D^3 B_1 B_2 B_3 \cdots$ is thus an ambient isotopy of $\mathbb{R}^3$, and it need not fix D^3.

Remarks 2.16. (1) Both the diffeomorphism type and the ambient isotopy type of any infinite concatenation of blocks $D^3 B_1 B_2 B_3 \cdots$ are determined by any tail $B_m B_{m+1} B_{m+2} \cdots$ of $B_1 B_2 B_3 \cdots$. This is a consequence of the fact that the multiray τ determined by $D^3 B_1 B_2 B_3 \cdots$ may be shrunk outwards. Using a

regular neighborhood $\nu\tau$ of τ in $\mathbb{R}^3$, one may construct an ambient isotopy of $\mathbb{R}^3$, with support in Int $\nu\tau$, that sends each component τ_k of τ into itself at all times and slides the initial point $\partial\tau_k$ up along τ_k any given finite amount.

(2) Let $f_i : B_i \to C_i$ be diffeomorphisms of blocks where $1 \leq i \leq k$ (a similar remark applies to the case $i \in \mathbb{Z}^+$). If each f_i is pointwise the identity on boundary 2-spheres, then evidently we get a diffeomorphism of the concatenations $B_1 B_2 \cdots B_k \to C_1 C_2 \cdots C_k$ by pasting together diffeomorphisms (and smoothing the resulting homeomorphism near boundary 2-spheres [14, p. 182]).

(3) If $B_1 \approx C_1$ and $B_2 \approx C_2$, then, in general, one may not conclude that $B_1 B_2$ and $C_1 C_2$ are diffeomorphic. For instance, let $f_1 = \mathrm{Id} : A \to A$, and let $f_2 : A \to \overline{A}^*$ be the diffeomorphism from Table 3 below; both diffeomorphisms preserve orientation, f_1 is pointwise the identity everywhere, and f_2 preserves boundary spheres setwise. However, by Table 4 below, $AA \not\approx A\overline{A}^*$.

We close this section with a special instance where one may conclude that $B_1 B_2 \approx C_1 C_2$.

Lemma 2.17. *Let $f : B \to \varepsilon_n$ be a diffeomorphism. Then, there is a trivial block ε'_n and diffeomorphisms $g, k : B \to \varepsilon'_n$ such that g is pointwise the identity on outer boundary 2-spheres and k is pointwise the identity on inner boundary 2-spheres.*

Proof. Without loss of generality, B and ε_n have underlying thickened sphere $S^2[1, 2]$. Composing f with inversion across $S^2[3/2]$ if necessary, we assume f preserves boundary 2-spheres setwise. Define $h : S^2[1, 2] \to S^2[1, 2]$ by:

$$h(p) = \frac{\|p\|}{2} f^{-1}\left(2\frac{p}{\|p\|}\right).$$

That is, $h = f^{-1}$ on $S^2[2]$, and h is extended radially to $S^2[1, 2]$. So, $\varepsilon'_n := h(\varepsilon_n)$ is a trivial block and $g := hf$ is the desired diffeomorphism. The other case is similar. $\qquad\square$

Lemma 2.18. *Let $B_1 \approx C_1$ and $B_2 \approx C_2$. Suppose one of the blocks B_1, B_2, C_1, or C_2 is diffeomorphic to ε_n. Then, $B_1 B_2 \approx C_1 C_2$.*

Proof. Without loss of generality, the four given blocks have underlying thickened spheres $S^2[1, 2]$. Assume there is a diffeomorphism $f : B_1 \to \varepsilon_n$ (the other cases are similar). By Lemma 2.17, there is a diffeomorphism $g : B_1 \to \varepsilon'_n$ that is pointwise the identity on $S^2[2]$. We have diffeomorphisms:

$$B_1 B_2 \to \varepsilon'_n B_2 \to B_2,$$

where the first is given by pasting together g and Id, and the second is given by Lemma 2.15. As $C_1 \approx B_1 \approx \varepsilon_n$, we similarly obtain diffeomorphisms:

$$C_1 C_2 \to \varepsilon''_n C_2 \to C_2.$$

As $B_2 \approx C_2$, the result follows. $\qquad\square$

3. Diffeomorphisms Between Blocks in $\mathcal{B}$

Let $\mathrm{Hom}(\mathcal{B})$ denote the set of all diffeomorphisms $h : B_1 \to B_2$ such that B_1 and B_2 lie in $\mathcal{B}$. The category $\mathcal{C}$ with objects $\mathcal{B}$ and with morphisms $\mathrm{Hom}(\mathcal{B})$ is a *groupoid* (i.e. every morphism is an isomorphism). The composition $h_2 \circ h_1$ of elements $h_1, h_2 \in \mathrm{Hom}(\mathcal{B})$ exists if and only if the codomain block of h_1 equals the domain block of h_2. Given a block in $\mathcal{B}$, we number its tangle components by 1, 2, 3, beginning with the component having boundary points on the x-axis and proceeding counterclockwise (CCW).

To each $h : B_1 \to B_2$ in $\mathrm{Hom}(\mathcal{B})$, we associate:

(1) The domain block B_1.
(2) The codomain block B_2.
(3) The *orientation character*: $+1$ if h preserves orientation of $S^2[1,2]$, and -1 otherwise.
(4) The *boundary character*: $+1$ if h preserves the boundary 2-spheres componentwise, and -1 if h swaps them.
(5) The *tangle permutation*: $\sigma = \sigma(h) \in \mathrm{Sym}(3)$, defined using the numbering of tangle components.

The *type* of h is the 5-tuple of these parameters. The set of possible types is:

$$U := \mathcal{B} \times \mathcal{B} \times \{\pm 1\} \times \{\pm 1\} \times \mathrm{Sym}(3).$$

The set U has cardinality 384 and has natural operations:

$$(B_1, B_2, \varepsilon, \delta, \sigma)^{-1} := (B_2, B_1, \varepsilon, \delta, \sigma^{-1}),$$

$$(B_2, B_3, \varepsilon_2, \delta_2, \sigma_2) \circ (B_1, B_2, \varepsilon_1, \delta_1, \sigma_1) := (B_1, B_3, \varepsilon_1\varepsilon_2, \delta_1\delta_2, \sigma_2\sigma_1).$$

The former is a unary operation defined on all types, and the latter is a partial composition. We compose permutations right to left, as with functions; MAGMA does the reverse. The category $\mathcal{D}$ with objects $\mathcal{B}$ and morphisms U is also a groupoid; if $B \in \mathcal{B}$, then the *identity morphism* is $1_B := (B, B, +1, +1, \mathrm{Id})$. We have a functor of groupoids $F : \mathcal{C} \to \mathcal{D}$. It is the identity on objects and is defined on morphisms by:

$$\mathrm{Hom}(\mathcal{B}) \xrightarrow{\ \ F\ \ } U$$
$$h \longmapsto \mathrm{type}(h).$$

We wish to determine the types realized by diffeomorphisms, i.e. the image of F on morphisms. Table 2 presents exactly these types, where $\mathrm{A}_3 := \langle (1,2,3) \rangle$ is the alternating group of degree 3 and $C := (1,2)\mathrm{A}_3$ is the coset of transpositions. The remainder of this section is devoted to proving Table 2.

Table 3 lists some obvious diffeomorphisms of blocks and their types. To see the last two diffeomorphisms, perform the rotation on the given domain block, then perform a simple ambient isotopy relative to boundary.

Table 2. Types of diffeomorphisms of blocks in $\mathcal{B}$. Columns indicate domain and orientation character. Rows indicate codomain and boundary character. A given subset of Sym(3) specifies tangle permutations realized by such a diffeomorphism. An empty entry means no such diffeomorphism exists.

		A		$\overline{A}$		A^*		$\overline{A}^*$	
		$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
A	$+1$	A_3			A_3		C	C	
	-1	C			C		A_3	A_3	
$\overline{A}$	$+1$		A_3	A_3		C			C
	-1		C	C		A_3			A_3
A^*	$+1$		C	C		A_3			A_3
	-1		A_3	A_3		C			C
$\overline{A}^*$	$+1$	C			C		A_3	A_3	
	-1	A_3			A_3		C	C	

Table 3. Diffeomorphisms of blocks.

Diffeomorphism	Domain/codomain	Type
Identity	$B \to B,\ B \in \mathcal{B}$	$(B, B, +1, +1, \mathrm{Id})$
CCW rotation by $2\pi/3$ about z-axis	$B \to B,\ B \in \mathcal{B}$	$(B, B, +1, +1, (1,2,3))$
Reflection across xy-plane	$A \to \overline{A}$ $A^* \to \overline{A}^*$	$(A, \overline{A}, -1, +1, \mathrm{Id})$ $(A^*, \overline{A}^*, -1, +1, \mathrm{Id})$
Inversion across intermediate S^2	$A \to A^*$ $\overline{A} \to \overline{A}^*$	$(A, A^*, -1, -1, \mathrm{Id})$ $(\overline{A}, \overline{A}^*, -1, -1, \mathrm{Id})$
Rotation by π about x-axis	$A \to \overline{A}^*$ $\overline{A} \to A^*$	$(A, \overline{A}^*, +1, +1, (2,3))$ $(\overline{A}, A^*, +1, +1, (2,3))$

The image of F is a subgroupoid of $\mathcal{D}$, with objects $\mathcal{B}$ and morphisms $F(\mathrm{Hom}(\mathcal{B}))$. In particular, $F(\mathrm{Hom}(\mathcal{B}))$ is closed under taking inverses and forming well-defined compositions. So: (i) take the types in Table 3, (ii) include their inverses, and (iii) further include all well-defined binary compositions. This yields the 96 realizable types in Table 2. It remains to exclude the other types.

Exclusion of types will utilize fundamental groups of certain tangle complements. If B is a block, or a concatenation of blocks, and τ is the tangle therein, then $B - \tau$ denotes the complement of τ in the thickened sphere. As explained in Sec. 2.2, we count classes of homomorphisms into small symmetric groups $\mathrm{Sym}(n)$ using MAGMA. Table 4 collects this data.

Lemma 3.1. *Let $B_1 \neq B_2$ be blocks in $\mathcal{B}$. Then, the type $(B_1, B_2, +1, +1, \mathrm{Id})$ is not in the image of F.*

Proof. Suppose otherwise. Then, there is a diffeomorphism $h : B_1 \to B_2$ of type $(B_1, B_2, +1, +1, \mathrm{Id})$.

Table 4.   Numbers of classes of homomorphisms into Sym(n).

	$\pi_1(AA - \tau)$	$\pi_1(A\overline{A} - \tau)$	$\pi_1(AA^* - \tau)$	$\pi_1(A\overline{A}^* - \tau)$
Sym(1)	1	1	1	1
Sym(2)	4	4	4	4
Sym(3)	11	11	11	11
Sym(4)	63	63	63	63
Sym(5)	342	342	354	330
Sym(6)	3111	3255	3525	3105

Let $S_{0,3}$ denote the 2-sphere S^2 with three marked points p_1, p_2, p_3. The mapping class group $\mathrm{Mod}(S_{0,3})$ is the group of orientation preserving diffeomorphisms of S^2 that send $\{p_1, p_2, p_3\} \to \{p_1, p_2, p_3\}$, modulo isotopies of S^2 fixing p_1, p_2, and p_3 at all times [9, pp. 44–45]. Recall that the natural map

$$\mathrm{Mod}(S_{0,3}) \xrightarrow{\ \cong\ } \mathrm{Sym}(3), \tag{3.1}$$

which sends an element of $\mathrm{Mod}(S_{0,3})$ to its action on the three marked points, is an isomorphism [9, Sec. 2.2.2].

By hypothesis, h preserves boundary 2-spheres componentwise and preserves orientation of S^2[1, 2]. So, $h \mid S^2$ is an orientation preserving diffeomorphism. As the tangle permutation of h is the identity, (3.1) permits us to assume h is the identity on inner boundary 2-spheres. Pasting together the diffeomorphisms $\mathrm{Id} : A \to A$ and $h : B_1 \to B_2$ yields a diffeomorphism $AB_1 \to AB_2$. This contradicts the last row of Table 4 since $B_1 \neq B_2$. $\qquad\square$

Consider two types $\alpha, \beta \in U$ such that $\beta \circ \alpha$ is defined. As the image of F is a subgroupoid of $\mathcal{D}$, if any two of α, β, or $\beta \circ \alpha$ lie in the image of F, then the third does as well. Therefore, if α is any type forbidden by Lemma 3.1, β is any of the already realized 96 types in the image of F, and $\beta \circ \alpha$ is defined in $\mathcal{D}$, then $\beta \circ \alpha$ is not in the image of F. A tedious, but completely straightforward calculation (facilitated by MAGMA), shows that this yields 288 types not in the image of F. This completes our proof of Table 2, and yields the following.

Corollary 3.2. *The blocks A, $\overline{A}$, A^*, and $\overline{A}^*$ are pairwise distinct up to ambient isotopy relative to boundary.*

4. Irreducible Blocks

This section introduces the notion of an *irreducible block*. Such blocks play a central role in our construction of Borromean rays. First, recall the following standard definitions and accompanying lemma.

A 2-sphere Σ in a thickened sphere is *essential* provided it does not bound a 3-disk in the thickened sphere. Otherwise, Σ is *inessential*. Let a be an arc transverse to a 2-manifold Σ in a 3-manifold. Then, $|a \cap \Sigma|$ denotes the number of points

in $a \cap \Sigma$ (ignoring any orientations). The *mod 2 intersection number* of a and Σ, denoted $\#_2(a, \Sigma)$, is $|a \cap \Sigma|$ mod 2.

Lemma 4.1. *Let a be a neatly embedded arc in $S^2[t_1, t_2]$. Assume a has one boundary point in $S^2[t_1]$ and the other in $S^2[t_2]$. Let Σ be a 2-sphere embedded in the interior of $S^2[t_1, t_2]$ and transverse to a. The following are equivalent: (i) Σ is essential in $S^2[t_1, t_2]$, (ii) $\#_2(a, \Sigma) \neq 0$, and (iii) there is a neighborhood U of $\partial S^2[t_1, t_2]$ in $S^2[t_1, t_2]$ and an ambient isotopy H_t, $0 \leq t \leq 1$, of $S^2[t_1, t_2]$ such that:*

$$H_0 = \mathrm{Id}. \tag{4.1}$$

$$H_t \,|\, U = \mathrm{Id} \ \textit{for all } 0 \leq t \leq 1. \tag{4.2}$$

$$H_1(\Sigma) \ \textit{is a level 2-sphere in } \mathrm{Int}\, S^2[t_1, t_2]. \tag{4.3}$$

Proof. Assume, without loss of generality, that $[t_1, t_2] = [1, 2]$. Let D denote $D^3 \cup S^2[1, 2]$, the 3-disk of radius 2. By the three-dimensional Schoenflies theorem [4, Chap. III; 20, Sec. 17; 12, Theorem 1.1], Σ bounds a unique 3-disk $\Delta \subset \mathrm{Int}\, D$. Let $Y := D - \mathrm{Int}\, \Delta$. So, $D = \Delta \cup Y$ and $\Delta \cap Y = \Sigma$.

Case 1: $D^3 \subset \mathrm{Int}\, Y$. Then, Δ lies in the interior of $S^2[1, 2]$. So, Σ is inessential and clearly $\#_2(a, \Sigma) = 0$.

Case 2: $D^3 \subset \mathrm{Int}\, \Delta$. Then, Σ cannot bound a 3-disk in $S^2[1, 2]$ and, hence, is essential in $S^2[1, 2]$. The arc a has one boundary point in $\mathrm{Int}\, \Delta$ and one outside Δ, so $\#_2(a, \Sigma) = 1$. Let Σ' be a level 2-sphere between Σ and $S^2[2]$. Let D' be the 3-disk in $\mathrm{Int}\, D$ with boundary Σ'. Let $R := D' - \mathrm{Int}\, \Delta$ be the compact region in $\mathrm{Int}\, D$ with boundary $\Sigma \sqcup \Sigma'$. By uniqueness of disk embeddings [14, p. 185], there is an ambient isotopy of D' carrying Δ to a round 3-disk. Hence, there is a diffeomorphism $g : R \to S^2 \times [0, 1]$ sending Σ to $S^2 \times \{0\}$. Construct a vector field v on $S^2[1, 2]$ as follows. On R, v is the pushforward by g^{-1} of the constant vector field $(0, 1)$ on $S^2 \times [0, 1]$. Extend v to the rest of $S^2[1, 2]$, making it 0 outside a small neighborhood of R. The isotopy generated by v is the desired H_t. $\qquad\square$

Next, we give two definitions of *irreducible block* and then we prove they are equivalent. Let $B = (S^2[t_1, t_2], \tau)$ be a block where τ has $n \geq 1$ components.

Definition 4.2 (Irreducible Block, First Definition). The block B is *irreducible* provided: if B is diffeomorphic to a concatenation of blocks $B_1 B_2$, then B_1 or B_2 (or both) is diffeomorphic to a trivial block ε_n.

Definition 4.3 (Irreducible Block, Second Definition). Let Σ be a 2-sphere embedded in the interior of $S^2[t_1, t_2]$ and transverse to τ. Assume Σ meets each component of τ at exactly one point. The block B is *irreducible* provided: there is a neighborhood U of $\partial S^2[t_1, t_2]$ in $S^2[t_1, t_2]$ and an ambient isotopy H_t, $0 \leq t \leq 1$, of $S^2[t_1, t_2]$ such that:

$$H_0 = \mathrm{Id}. \tag{4.4}$$

$$H_t \,|\, U = \mathrm{Id} \text{ for all } 0 \le t \le 1. \tag{4.5}$$

$$H_t \text{ fixes } \tau \text{ setwise for all } 0 \le t \le 1. \tag{4.6}$$

$$H_1(\Sigma) \text{ is a level 2-sphere in } S^2[t_1, t_2]. \tag{4.7}$$

$$\eta \,|\, \tau \text{ has no critical points between } H_1(\Sigma) \text{ and } S^2[t_1],$$
$$\text{or between } H_1(\Sigma) \text{ and } S^2[t_2]. \tag{4.8}$$

Proposition 4.4. *The two definitions of irreducible block are equivalent.*

Proof. Assume B is irreducible according to the second definition. Suppose $h : B \to B_1 B_2$ is a diffeomorphism. Assume h sends the inner boundary 2-sphere to the inner boundary 2-sphere, the other case being similar. Let Σ' be the level 2-sphere in $B_1 B_2$ along which the concatenation takes place. Let $\Sigma := h^{-1}(\Sigma')$. Then, Σ satisfies the hypotheses of the second definition. Let H_t be the isotopy provided by the second definition. So, $H_1(\Sigma)$ is a level 2-sphere in $S^2[t_1, t_2]$ and, say, $\eta \,|\, \tau$ has no critical points in the compact region R with boundary $H_1(\Sigma) \sqcup S^2[t_1]$. By Corollary 2.10, the block $(R, \tau \cap R)$ is diffeomorphic to a trivial block ε_n. Hence, $B_1 \approx \varepsilon_n$, as desired.

Next, assume B is irreducible according to the first definition. Let Σ satisfy the hypotheses of the second definition. By Lemma 4.1, Σ is essential in $S^2[t_1, t_2]$, and there is an isotopy of $S^2[t_1, t_2]$ (probably disturbing τ) which carries Σ to a level 2-sphere, Σ'. Let τ' be the image of τ under this isotopy. This Σ' divides $(S^2[t_1, t_2], \tau')$ into two obvious blocks B_1 and B_2. Evidently, B is diffeomorphic to the concatenation $B_1 B_2$. The first definition of irreducible block implies that, say, B_1 is diffeomorphic to ε_n. By Lemma 2.5 (Trivial Block Uniqueness), we can and do assume ε_n has underlying thickened sphere $S^2[1, 2]$. Let R denote the compact region in $S^2[t_1, t_2]$ between Σ and $S^2[t_1]$. Hence, there is a diffeomorphism of pairs $g : (R, \tau \cap R) \to \varepsilon_n$, and g sends $S^2[t_1]$ to S^2. Let $\Sigma'' := S^2[t_1 + \alpha]$ for some small $\alpha > 0$ to be specified. First, choose α small enough so Σ'' lies between Σ and $S^2[t_1]$. Next, reduce α if necessary so that:

$$\eta \,|\, \tau \text{ has no critical points on or between } \Sigma'' \text{ and } S^2[t_1]. \tag{4.9}$$

This reduction is possible since τ is neatly embedded. Note that further reducing α maintains condition (4.9). Finally, reduce α if necessary so that:

$$g(\Sigma'') \text{ is nowhere tangent to any radial arc in } S^2[1, 2]. \tag{4.10}$$

This last reduction is possible since: (i) g is a diffeomorphism, (ii) $g(S^2[t_1]) = S^2$, and (iii) $S^2[t_1]$ is compact. As $g(\Sigma'')$ is essential in $S^2[1, 2]$, condition (4.10) implies that:

$$\text{Each radial arc in } S^2[1, 2] \text{ of length 1 intersects } g(\Sigma'') \text{ in exactly}$$
$$\text{one point and transversely.} \tag{4.11}$$

Condition (4.11) permits construction of an ambient isotopy of $S^2[1,2]$ that carries $g(\Sigma'')$ to $S^2[3/2]$ and merely slides points along radial arcs. So, by an abuse of notation, we further assume the diffeomorphism of pairs g itself sends Σ'' to $S^2[3/2]$. Let R' be the compact region in $S^2[t_1, t_2]$ between Σ and Σ''. Construct a vector field v on $S^2[t_1, t_2]$ as follows. On R', v is the pushforward by g^{-1} of the vector field $-p/(2\|p\|)$ on $S^2[3/2, 2]$. Note that v is tangent to τ on R'. Extend v to the rest of $S^2[t_1, t_2]$, making it 0 outside a small neighborhood of R' and ensuring tangency to τ. The isotopy generated by v is the desired H_t. □

In general, it appears to be a difficult problem to decide whether a given block is irreducible. We prove next that trivial blocks are irreducible and then observe some corollaries.

Proposition 4.5. *Each trivial block $\varepsilon_n = (S^2[t_1, t_2], \tau)$, $n \geq 1$, is irreducible.*

Proof. Let Π denote the annulus where $S^2[1,2]$ meets the xy-plane. Lemma 2.5 reduces us to the case where ε_n has underlying thickened sphere $S^2[1,2]$ and τ consists of n equally spaced radial arcs in Π. We prove ε_n is irreducible according to the first definition. By the first paragraph of the proof of Proposition 4.4, it suffices to consider a 2-sphere, Σ, embedded in the interior of $S^2[1,2]$, transverse to τ, and intersecting each component τ_i of τ in one point p_i. Let $X \subset S^2[1,2]$ be the compact set between Σ and S^2. It suffices to produce a diffeomorphism $g : (X, X \cap \tau) \to \varepsilon_n$.

We will improve Σ (and, hence, X) by ambient isotopies of $S^2[1,2]$. Improved spaces will be denoted by their original names, except Π always denotes $S^2[1,2] \cap (xy\text{-plane})$. So, assume Σ intersects Π transversely. Thus, $\Sigma \cap \Pi$ is a closed 1-manifold and one component, K, of $\Sigma \cap \Pi$ must contain all of the points p_i.

If $\Sigma \cap \Pi \neq K$, then consider a component, C, of $\Sigma \cap \Pi$ that is innermost in its component of $\Sigma - K$. Let D_1 be the 2-disk in $\Sigma - K$ with boundary C. Let D_2 be the 2-disk in Π with boundary C. Then, $D_1 \cup D_2$ is an embedded 2-sphere in the interior of $S^2[1,2]$ disjoint from τ. By Lemma 4.1, $D_1 \cup D_2$ is inessential in $S^2[1,2]$. Let D be the 3-disk in $S^2[1,2]$ with boundary $D_1 \cup D_2$. This D permits construction of an isotopy of $S^2[1,2]$, with support near D, that carries D_1 past D_2 to a parallel copy of D_2. Thus, C (at least) has been eliminated from $\Sigma \cap \Pi$. Repeating this operation finitely many times, we get $\Sigma \cap \Pi = K$.

Now, we give a bootstrapping definition of the required diffeomorphism g. First, g sends $S^2 \to S^2$ by the identity. Second, g sends $X \cap \tau_i$ to τ_i by an affine diffeomorphism for each i. Third, g sends K to $S^2[2] \cap \Pi$. Fourth, g sends a smooth, regular neighborhood of $(S^2 \cap \Pi) \cup (X \cap \tau) \cup K$ in $X \cap \Pi$ to a smooth, regular neighborhood of $(S^2 \cap \Pi) \cup \tau \cup (S^2[2] \cap \Pi)$ in Π. This step may be accomplished, quite concretely, by judiciously choosing (closed) collars [14, Sec. 4.6] and ambiently rounding corners. Fifth, g sends $X \cap \Pi$ to Π. By the smooth 2-dimensional Schoenflies theorem [3, Remark 9.19], this step evidently requires extension of g over n smooth 2-disks

$\Delta_i \subset X \cap \Pi$, $i = 1, \ldots, n$. Let $R_i \subset \Pi$ denote the 2-disk with boundary $g(\partial \Delta_i)$. As every diffeomorphism of S^1 extends to one of D^2, each diffeomorphism $g \,|\, \partial \Delta_i$ extends to a diffeomorphism $\psi_i : \Delta_i \to R_i$. Extending g over Δ_i by ψ_i yields a well-defined homeomorphism (smooth except possibly at $\partial \Delta_i$). By an isotopy of ψ_i, relative to $\partial \Delta_i$ and with support in a collar of $\partial \Delta_i$ [14, p. 182], this extension is a diffeomorphism. Sixth, g sends $\Sigma \to S^2[2]$. Seventh, g sends a smooth, regular neighborhood of $S^2 \cup (X \cap \Pi) \cup \Sigma$ in X to a smooth, regular neighborhood of $S^2 \cup \Pi \cup S^2[2]$ in $S^2[1,2]$. This is done as in step four (product a nice corner rounding with S^1). Finally, g sends X to $S^2[1,2]$. By the smooth 3-dimensional Schoenflies theorem, this step requires extension of g over two smooth 3-disks. This is done as in step five, except using the fact that every diffeomorphism of S^2 extends to one of D^3 [21; 27; 29, pp. 202–206]. By construction, the diffeomorphism $g : X \to S^2[1,2]$ sends $X \cap \tau$ to τ. □

Corollary 4.6. *Let B_i, $1 \le i \le k$, be any blocks. If $B_1 B_2 \cdots B_k \approx \varepsilon_n$, then $B_i \approx \varepsilon_n$ for each $1 \le i \le k$.*

Proof. By induction, it suffices to consider the case $k = 2$. The case $k = 2$ follows from the proof of Proposition 4.5, since X can be the compact region between Σ and S^2, or between Σ and $S^2[2]$. Alternatively, it is instructive to see that the case $k = 2$ follows from the statement of Proposition 4.5 as follows. Proposition 4.5 implies that $B_1 \approx \varepsilon_n$ or $B_2 \approx \varepsilon_n$. Assume $B_1 \approx \varepsilon_n$ (the other case is similar). We have diffeomorphisms:

$$\varepsilon_n \approx B_1 B_2 \approx \varepsilon_n B_2 \approx B_2,$$

where the first exists by hypothesis, the second follows from Lemma 2.18 since $B_1 \approx \varepsilon_n$ and $B_2 \approx B_2$, and the last is given by Lemma 2.15. □

Corollary 4.7. *Let B_i, $i \in \mathbb{Z}^+$, be any blocks. If $B_1 B_2 B_3 \cdots \approx \varepsilon_n \varepsilon_n \varepsilon_n \cdots$, then $B_i \approx \varepsilon_n$ for each $i \ge 1$.*

Proof. Let $f : B_1 B_2 B_3 \cdots \to \varepsilon_n \varepsilon_n \varepsilon_n \cdots$ be a diffeomorphism (of pairs). Let τ and r be the n-component multirays determined by these concatenations respectively. Clearly, r is radial. Isotopies of f will send τ to r at all times. Note that $f(S^2) = S^2$ since f must restrict to a diffeomorphism on the boundaries of the total spaces. Let $\Sigma := f(S^2[2])$. By a radial isotopy, relative to S^2, we may assume $\Sigma \subset \mathrm{Int}\, S^2[1,2]$. This Σ satisfies the hypotheses in the second definition of irreducible block. Propositions 4.4 and 4.5 permit us to isotop f, relative to S^2, so that $\Sigma = S^2[2]$. Having isotoped f so that $f(S^2[i]) = S^2[i]$ for $1 \le i \le k$ and some $k \ge 2$, the same argument permits us to further isotop f, relative to $S^2[1,k]$, so that $f(S^2[k+1]) = S^2[k+1]$. Evidently, the composition of all of these (infinitely many) isotopies is a well-defined, smooth, proper isotopy. So, we can and do assume $f(S^2[k]) = S^2[k]$ for all $k \in \mathbb{Z}^+$. The result is now immediate. □

Corollary 4.8. *Let B_i, $i \in \mathbb{Z}^+$, be any blocks. If $D^3 B_1 B_2 B_3 \cdots \approx D^3 \varepsilon_n \varepsilon_n \varepsilon_n \cdots$, then $B_i \approx \varepsilon_n$ for all sufficiently large $i \in \mathbb{Z}^+$.*

Proof. Let $f : D^3 B_1 B_2 B_3 \cdots \to D^3 \varepsilon_n \varepsilon_n \varepsilon_n \cdots$ be a diffeomorphism (of pairs). Let τ and r be the n-component multirays determined by these concatenations respectively. Isotopies of f will send τ to r at all times. By compactness, there exists $k \geq 2$ such that $f(S^2[k])$ is disjoint from D^3. As in the previous proof, we may isotop f, relative to D^3, so that $f(S^2[k]) = S^2[k]$. Restricting f to $S^2[k, \infty)$ implies $B_k B_{k+1} B_{k+2} \cdots \approx \varepsilon_n \varepsilon_n \varepsilon_n \cdots$. Now, apply the previous corollary. $\qquad\square$

Borromean blocks were defined in Sec. 2.3. A multiray $\tau \subset \mathbb{R}^3$ forms *Borromean rays* provided: (i) no diffeomorphism of $\mathbb{R}^3$ carries τ to a radial multiray, and (ii) each multiray obtained from τ by forgetting one component is ambiently isotopic to a two-component, radial multiray.

Corollary 4.9. *Let B_i, $i \in \mathbb{Z}^+$, be Borromean blocks and let*

$$(\mathbb{R}^3, \tau) = D^3 B_1 B_2 B_3 \cdots .$$

Then, τ forms Borromean rays. In particular, the conclusion holds if each $B_i \in \mathcal{B}$.

Proof of Corollary 4.9. By definition of Borromean block, $B_i \not\approx \varepsilon_3$. Corollary 4.8 implies that $D^3 B_1 B_2 B_3 \cdots \not\approx D^3 \varepsilon_3 \varepsilon_3 \varepsilon_3 \cdots$. Hence, no diffeomorphism of $\mathbb{R}^3$ carries τ to a radial multiray. Next, let σ be obtained from τ by forgetting any one component. Let C_i be obtained from B_i by forgetting the corresponding tangle component. By definition of Borromean block, each C_i is ambient isotopic (relative to boundary) to ε_2. Performing these isotopies, for $i \in \mathbb{Z}^+$, simultaneously yields an ambient isotopy of $\mathbb{R}^3$ carrying σ to a radial multiray. Thus, τ forms Borromean rays. Lastly, blocks in $\mathcal{B}$ are Borromean by Corollary 2.7. $\qquad\square$

Remarks 4.10. (1) Corollary 4.8 reduces the *infinite* problem of constructing knotted multirays to the *finite* problem of constructing nontrivial blocks. For instance, the infinitely generated group theory in [5] may be replaced by finitely generated group theory (as used in Sec. 2.2 above).

(2) The converse of Corollary 4.8 holds by Remarks 2.16(1).

We close this section by constructing infinitely many irreducible blocks containing two-component tangles. Recall the notion of a knot block from Example 2.3.

Lemma 4.11. *If $k \subset S^3$ is a prime knot, then $B(k)$ is an irreducible block.*

Proof. Let $B(k) = (S^2[1, 2], \tau)$ where τ_2 is radial and τ_1 contains the diagram K for k as in Fig. 6. Let Σ be a 2-sphere embedded in the interior of $S^2[1, 2]$, transverse to τ, and intersecting each component τ_i of τ in one point p_i. Perturb Σ so it coincides with the level sphere through p_2 near τ_2. All isotopies will be ambient
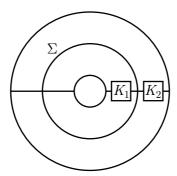
Fig. 10.   Block $B(k)$ after ambient isotopy carrying Σ to a level sphere.

and relative to a neighborhood of both $\partial S^2[1,2]$ and τ_2. Subsets that move will be called by their original names. As in the proof of Proposition 4.5, we isotop Σ to the level sphere containing p_2. Push τ_1 away from τ_2 by integrating a vector field tangent to level spheres. The result is shown in Fig. 10. As k is prime, one of the diagrams K_1 or K_2 must be trivial. $\qquad\square$

Corollary 4.12. *There exists a countably infinite collection of irreducible knot blocks, pairwise distinct up to diffeomorphism.*

Proof. Let $\mathcal{T}$ denote the set of torus knots $\mathfrak{t}(a,b)$ where $a > b \geq 2$ and $\gcd(a,b) = 1$. As torus knots are prime [2, p. 95], Lemma 4.11 implies that each $B(k)$, $k \in \mathcal{T}$, is irreducible. The fundamental groups of these torus knots are pairwise nonisomorphic [2, p. 47]. By Example 2.3, these knot blocks are pairwise distinct up to diffeomorphism. $\qquad\square$

Remark 4.13. Under concatenation, knot blocks commute, unlike distinct blocks in $\mathcal{B}$ (recall Table 4). Concatenating infinitely many knot blocks yields a multiray in $\mathbb{R}^3$ known as *Wilder rays*. They were classified by Fox and Harrold [11].

5. Unknotted Ball-Arc Pairs

This section identifies some unknotted ball-arc pairs in blocks. These tools will be used in the next two sections. Recall that a *ball-arc pair* is a pair (Δ, a) such that $a \approx D^1$ is neatly embedded in $\Delta \approx D^3$. Such a pair is *unknotted* provided it is diffeomorphic to the standard pair $(D^3, \{(0,0)\} \times D^1)$, and otherwise it is *knotted*. If $k \subset S^3$ is a smooth knot (not the unknot) and $(D, b) \subset (S^3, k)$ is an unknotted ball-arc pair such that $D \cap k = b$, then $(S^3 - \text{Int}\, D, k - \text{Int}\, b)$ is a knotted ball-arc pair. Every knotted ball-arc pair arises this way up to diffeomorphism.

Lemma 5.1. *Let $\varepsilon_2 = (S^2[1,2], \tau)$. Let Σ be a 2-sphere embedded in the interior of $S^2[1,2]$ and transverse to τ. Assume Σ meets τ at exactly two points p and q,*

both of which lie on one component of τ, say τ_1. Then, Σ bounds a 3-disk, Δ, in $S^2[1,2]$ and $(\Delta, \Delta \cap \tau)$ is an unknotted ball-arc pair.

Proof. By Lemma 4.1, Σ is inessential in $S^2[1,2]$. Let Π denote the annulus where $S^2[1,2]$ meets the xy-plane. Without loss of generality, $\tau \subset \Pi$ and Σ is transverse to Π. Thus, $\Sigma \cap \Pi$ is a closed 1-manifold and one component, K, of $\Sigma \cap \Pi$ must contain p and q; this is where the hypothesis $\Sigma \cap \tau_2 = \emptyset$ is used. As in the proof of Proposition 4.5 (paragraph three), we may arrange that $\Sigma \cap \Pi = K$. Now, it is straightforward to construct the required diffeomorphism (cf. paragraph four of the proof of Proposition 4.5). $\qquad\square$

Remarks 5.2. (1) Lemma 5.1 becomes false without the hypothesis $\Sigma \cap \tau_2 = \emptyset$ (i.e. with ε_2 replaced by ε_1). To see this, consider the block $B = (S^2[1,2], \tau')$ in Fig. 11 (left). The indicated sphere Σ' meets τ' in two points and bounds the 3-ball Δ'. Let K be any crossing diagram such that $(\Delta', \Delta' \cap \tau')$ is a knotted ball-arc pair. Straighten τ' using the argument in Example 2.1. Let τ, Σ, and Δ denote the respective images of τ', Σ', and Δ' under this ambient isotopy. Then, Σ is a 2-sphere in ε_1 meeting τ in exactly two points and transversely. However, $(\Delta, \Delta \cap \tau) \approx (\Delta', \Delta' \cap \tau')$ is a knotted ball-arc pair.

(2) Lemma 5.1 and the previous remark may be recast in $\mathbb{R}^3$ as follows. Consider a 2-sphere $\Sigma \subset \mathbb{R}^3$. Let $\Delta \subset \mathbb{R}^3$ be the 3-disk with $\partial \Delta = \Sigma$. Suppose a is a straight arc in $\mathbb{R}^3$ that is neatly embedded in Δ. Let $l \subset \mathbb{R}^3$ be the straight line containing a. If Σ is disjoint from $l - a$, then (Δ, a) is an unknotted ball-arc pair. If Σ meets $l - a$, then (Δ, a) may be a knotted ball-arc pair. In fact, every knotted ball-arc pair (Δ', a') in $\mathbb{R}^3$ is ambient isotopic to some such (Δ, a).

Proof. (i) Straighten a' near an endpoint q', (ii) let $q'' \neq q'$ be a point in the straightened end of a', and (iii) ambiently isotop the other endpoint of a' along a' until it coincides with q'' (cf. Fig. 11 (right)). $\qquad\square$
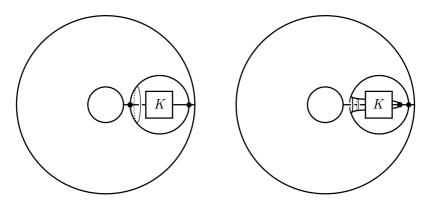

Fig. 11.   Sphere Σ' meeting the tangle τ' at two points in the block $B \approx \varepsilon_1$. At right is the result of an ambient isotopy that fixes τ' setwise.

Lemma 5.3. *Let a be a neatly embedded arc in $S^2[t_1, t_2]$. Assume a has one boundary point in $S^2[t_1]$ and the other in $S^2[t_2]$. Let D_1 be a 2-disk embedded in the interior of $S^2[t_1, t_2]$. Assume that $C := \partial D_1$ lies in some $S^2[t]$, C is disjoint from a, $\mathrm{Int}\, D_1$ is disjoint from $S^2[t]$, and D_1 is transverse to $S^2[t]$. Let D_2 and D_2' be the two 2-disks in $S^2[t]$ bounded by C. Then:*

$$\text{The intersection numbers } \#_2(a, D_2) \text{ and } \#_2(a, D_2') \text{ are unequal.} \qquad (5.1)$$

$$\text{The intersection numbers } \#_2(a, D_1) \text{ and } \#_2(a, D_2) \text{ are equal}$$
$$(\text{after possibly interchanging the names of } D_2 \text{ and } D_2'). \qquad (5.2)$$

$$\text{The sphere } D_1 \cup D_2 \text{ is inessential in } S^2[t_1, t_2]. \qquad (5.3)$$

$$\text{The sphere } D_1 \cup D_2' \text{ is essential in } S^2[t_1, t_2]. \qquad (5.4)$$

Proof. Immediate by Lemma 4.1. □

Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ denote closed upper half-space. The closed upper half-disk is $D^n_+ := D^n \cap \mathbb{R}^n_+$.

Lemma 5.4. *Let $\varepsilon_2 = (S^2[1, 2], \tau)$. Let τ_1 be a component of τ and let $q = \tau_1 \cap S^2[2]$. Suppose D_1 is a 2-disk neatly embedded in $S^2[1, 2]$ such that: (i) D_1 is transverse to τ, (ii) D_1 meets τ at one point $p \in \mathrm{Int}\,\tau_1$, and (iii) $C = \partial D_1$ lies in $S^2[2]$. Let D_2 be the 2-disk in $S^2[2]$ with boundary C and containing q. Then, $D_1 \cup D_2$ bounds a piecewise smooth 3-disk $D \subset S^2[1, 2]$ and $D \cap \tau_2 = \emptyset$. Further, there is a diffeomorphism of pairs $g : (D, D \cap \tau) \to (D^3_+, \{(0, 0)\} \times D^1_+)$ that sends D_1 to the upper hemisphere and D_2 to $D^2 \times \{0\}$.*
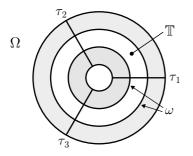
Proof. By Lemma 5.3 with $a = \tau_1$, $D_1 \cup D_2$ is inessential in $S^2[1, 2]$. By hypothesis, $D_1 \cap \tau_2 = \emptyset$. So, $D_2 \cap \tau_2 = \emptyset$, and $D \cap \tau_2 = \emptyset$ as well. The required diffeomorphism g is constructed in bootstrapping fashion (cf. paragraph four of the proof of Proposition 4.5): define g on ∂D, extend to a smooth, regular neighborhood of ∂D in D, and extend to the rest of D utilizing Lemma 5.1. □

6. Irreducibility of Blocks in $\mathcal{B}$

Theorem 6.1. *Each block in $\mathcal{B}$ is irreducible.*

The remainder of this section is devoted to proving A is irreducible, which suffices to prove Theorem 6.1. Recall the block $A = (S^2[1, 2], T)$ from Fig. 1. The (general position) immersion f yielding A (see Fig. 2) plays a central role in our proof. We pause to explain f and fix some notation. We assume the reader has Fig. 2 at hand.

The domain of f is $\Omega := S^1[1, 2] \subset \mathbb{R}^2$. The compact annulus Ω contains three equally spaced radial arcs, τ_1, τ_2, and τ_3, as in Fig. 12. Let $\tau = \tau_1 \cup \tau_2 \cup \tau_3$. Note that $f(\tau) = T$, the tangle in A. We let $\mathbb{T} := S^1[4/3, 5/3]$, the subannulus of Ω that
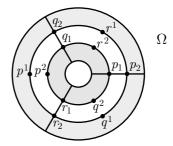
Fig. 12. Two labelings of the domain Ω of the immersion f.

is unshaded in Fig. 12. The boundary of $\mathbb{T}$ is $\omega = \partial \mathbb{T}$, the disjoint union of $S^1[4/3]$ and $S^1[5/3]$.

Given a subset $X \subset \Omega$, it will be convenient to let X' denote $f(X)$. (A notable exception is $T = f(\tau)$.) In particular, $\Omega' = f(\Omega)$, $\mathbb{T}' = f(\mathbb{T})$, and $\omega' = f(\omega)$.

For each $t \in [1, 2]$, $f \mid S^1[t]$ is an embedding, namely the composition of: a rigid rotation, a homothety, and a translation in the z-direction. We will see that:

(1) The multiple points of f are double points where $f(S^1[4/3]) = f(S^1[5/3]) = \omega'$.
(2) $\omega' = S^1[5/3] \times \{\varepsilon\}$ where $0 < \varepsilon \ll 1$.
(3) $\mathbb{T}'$ is a torus in $S^2[1, 2]$, smooth except for corners along ω'.

On $\partial \Omega$, f is inclusion $\iota : (x, y) \mapsto (x, y, 0)$. Figure 13 shows the intersection of Ω' with any plane in $\mathbb{R}^3$ containing the z-axis. On each of the three subannuli of Ω in Fig. 12, f is defined as follows. The annulus $S^1[1, 4/3]$ is stretched radially to $S^1[1, 5/3]$, then the outer boundary component is twisted by π radians CCW while fixing the inner boundary component, then level circles near the outer boundary component are lifted up a bit in the z-direction (to yield general position). The annulus $S^1[5/3, 2]$ maps into $S^2[1, 2]$ by ι, then level circles near the inner boundary component are lifted up a bit in the z-direction (again, to yield general position). Finally, f is defined on $\mathbb{T}$, interpolating $f \mid S^1[1, 4/3]$ and $f \mid S^1[5/3, 2]$, so as to yield a torus $\mathbb{T}' = f(\mathbb{T})$ as in Figs. 2 and 13. The two components of ω are identified under f after half a rotation of $S^1[4/3]$. This completes our description of f.

For distinct $i, j \in \{1, 2, 3\}$, let $S_{i,j}$ denote the closed sector in Ω between τ_i and τ_j of angular measure $2\pi/3$. Note that $f \mid S_{i,j}$ is an embedding. Fix distinct $i, j, k \in \{1, 2, 3\}$. Observe that $T_k = f(\tau_k)$ meets $f(S_{i,j})$ (transversely) at exactly two points. For example, using the labelings in Fig. 12, T_1 meets $f(S_{2,3})$ at the two points:

$$f(p_1) = f(p^1) \quad \text{and} \quad f(p_2) = f(p^2).$$

Similarly, $f(q_i) = f(q^i)$ and $f(r_i) = f(r^i)$ for $i = 1$ and 2. The points p^i, q^i, and r^i, where $i = 1$ and 2, will be referred to as *special points*.
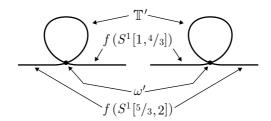
Fig. 13.   Intersection of $\Omega' = f(\Omega)$ with plane containing z-axis.

We prove A is irreducible according to Definition 4.2. It suffices to consider a 2-sphere, Σ, embedded in Int $S^2[1,2]$, transverse to T, and meeting each component of T at exactly one point. We improve Σ by ambient isotopies of $S^2[1,2]$ that fix T setwise at all times. By an abuse, we refer to each improved Σ as Σ. We view $\Omega' = f(\Omega)$ as an auxiliary object, unaffected by these isotopies. Perturb Σ so that $\Sigma \cap T$ is disjoint from ω'. Perturb Σ again so that further Σ meets Ω' in general position. In particular, $\Sigma \cap \Omega'$ is an immersed, closed 1-manifold in Σ in general position. Define:

$$\sigma := f^{-1}(\Sigma) = f^{-1}(\Sigma \cap \Omega') \subset \Omega,$$

which is an embedded, closed 1-manifold in Int Ω, transverse to ω and τ. Each component, τ_i, of τ meets σ at exactly one point (not in ω). So, there exists one component, K, of σ that meets each τ_i at one point (transversely) and K is essential in Ω.

Claim 6.2. It suffices to arrange that $\sigma = K$ and $K \cap \mathbb{T} = \emptyset$.

Proof. Similar to the argument in paragraph four of the proof of Proposition 4.5. $\square$

We give three operations for improving Σ. Define the *complexity* of Σ to be:

$$c(\Sigma) := |\omega \cap \sigma| + (\# \text{ of components of } \sigma) \in \mathbb{Z}^+.$$

Lemma 6.3. *Suppose that $a \subset \sigma$ and $b \subset \omega$ are arcs, and $a \cup b$ is a simple closed curve bounding a disk $D \subset \Omega$. Assume that D contains no special points, $D \cap \omega = b$, and Int $D \cap \sigma = \emptyset$ (see Fig. 14). Then, the points of ∂a can be eliminated from $\sigma \cap \omega$, and $c(\Sigma)$ decreases by at least 3.*

Proof. As D intersects only one component of ω, $f \,|\, D$ is an embedding. The disk $D' = f(D)$ permits construction of an isotopy of Σ, with support near D', that carries $f(a)$ past $f(b)$ to a parallel copy of $f(b)$. If D' intersects T, then $(D, D \cap \tau)$ is a disk-arc pair (all of which are unknotted). So, $(D', D' \cap T)$ is an unknotted disk-arc pair, and the isotopy fixes T setwise. The reduction in $c(\Sigma)$ follows from Fig. 15.
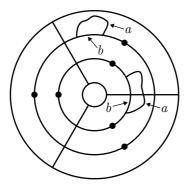
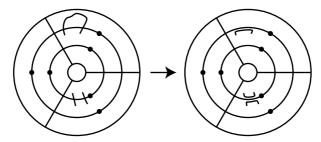Fig. 14.   Two pairs of arcs satisfying the hypotheses of Lemma 6.3.

Fig. 15.   Instance of first operation. The points ∂a of $\sigma \cap S^1[5/3]$ correspond to two points of $\sigma \cap S^1[4/3]$.

Four points of $\sigma \cap \omega$ are eliminated, and, at worst, the number of components of σ increases by one. □

Lemma 6.4. *Among the components of σ that are inessential in Ω and disjoint from ω, let C be one that is innermost in Ω. Then, C can be eliminated from σ, and $c(\Sigma)$ decreases by at least 1.*

Proof. Let $D \subset \Omega$ be the 2-disk with $\partial D = C$. Note that D is disjoint from ω and τ. So, $f \,|\, D$ is an embedding and $D' = f(D)$ is a 2-disk disjoint from T and bounding $C' = f(C)$. The circle C' bounds two 2-disks, D_1 and D_2, in Σ. The arc T_1 meets Σ at one point. So, without loss of generality, T_1 meets D_1 at one point (transversely, and in Int D_1), and $T_1 \cap D_2 = \emptyset$. By hypothesis, Int $D \cap \sigma = \emptyset$. So, $D' \cup D_1$ and $D' \cup D_2$ are embedded 2-spheres. By Lemma 4.1, $D' \cup D_1$ is essential in $S^2[1,2]$ and $D' \cup D_2$ is inessential. So, $D' \cup D_2$ bounds an embedded 3-disk $\Delta \subset S^2[1,2]$, and $\Delta \cap T = \emptyset$. The 3-disk Δ permits construction of an isotopy of Σ, with support near Δ, that carries D_2 past D' to a parallel copy of D'. □

Lemma 6.5. *Let C be a component of σ that bounds a 2-disk $D \subset \Omega$. Assume that $z := D \cap \omega$ is a neatly embedded arc in D, D contains exactly one special point x,*
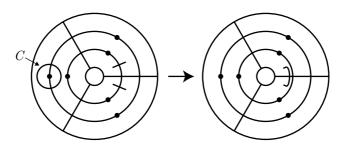
Fig. 16.   Instance of third operation. The points $C \cap \omega$ correspond to two points of $\sigma \cap S^1[4/3]$.

$x \in \operatorname{Int} z$, *and* $\operatorname{Int} D \cap \sigma = \emptyset$. *Then,* C *can be eliminated from* σ, *and* $c(\Sigma)$ *decreases by at least* 5.

Proof. Without loss of generality, $x = p^1$ as in Fig. 16. The embedded 2-disk $D' = f(D)$ meets T at exactly $f(p^1) = f(p_1) \in T_1$. The circle $C' = f(C)$ bounds two 2-disks, D_1 and D_2, in Σ. Without loss of generality, $T_1 \cap D_1 = \emptyset$ and T_1 meets D_2 at one point (transversely). By Lemma 4.1, $D' \cup D_2$ bounds an embedded 3-disk $\Delta \subset \operatorname{Int} S^2[1,2]$ and $\Delta \cap T \subset T_1$ is a neatly embedded arc in Δ. By Lemma 5.1, $(\Delta, \Delta \cap T)$ is an unknotted ball-arc pair. The pair $(\Delta, \Delta \cap T)$ permits construction of an isotopy of Σ, with support near Δ, that carries D_2 past D' to a parallel copy of D'. This isotopy fixes T_1 setwise and fixes T_2 and T_3 pointwise. $\square$

Improve Σ by applying Lemmas 6.3, 6.4, and 6.5 *in any order and as long as possible.* This is a finite process since the initial complexity of Σ is a positive integer and each operation strictly reduces the complexity. The complexity of the resulting improved Σ is $c(\Sigma) \in \mathbb{Z}^+$. The rest of this section shows that $c(\Sigma) = 1$ and $K \cap \mathbb{T} = \emptyset$, which suffices to prove Theorem 6.1 by Claim 6.2.

Lemma 6.6. *There do not exist arcs* $a \subset \sigma$ *and* $b \subset \omega$ *such that* $a \cup b$ *is a simple closed curve bounding a disk* $D \subset \Omega$ *where* D *is disjoint from the special points.*

Proof. We have $D \cap \omega = b \sqcup B$ where B is a finite disjoint union of neatly embedded arcs in D. We claim that $\operatorname{Int} D$ contains no closed component of σ. Otherwise, let $\Delta \subset \operatorname{Int} D$ be the 2-disk bounded by an innermost such component C. If $C \cap \omega = \emptyset$, then Lemma 6.4 applies to C, a contradiction. Thus, $\Delta \cap \omega$ is a nonempty finite disjoint union of neatly embedded arcs in Δ. Let b_0 be an arc of $\Delta \cap \omega$ that is outermost in Δ in the sense that ∂b_0 subtends an arc $a_0 \subset \partial \Delta$ such that:

$$a_0 \cap \omega = \partial a_0 = \partial b_0.$$

Lemma 6.3 applies to a_0 and b_0, a contradiction. The proof of the claim is complete.

If $B = \emptyset$, then Lemma 6.3 applies to a and b, a contradiction. Otherwise, there exists a component b_0 of B that is outermost in D in the sense that ∂b_0 subtends

an arc $a_0 \subset \partial D$ such that:

$$a_0 \cap B = \partial a_0 = \partial b_0.$$

Lemma 6.3 applies to a_0 and b_0, a contradiction. $\qquad\square$

Next, we show that σ contains no component inessential in Ω. Suppose, by way of contradiction, that σ contains component(s) inessential in Ω. Among these components, there must be one, call it C, that is innermost in Ω. By Lemma 6.4, C meets ω. So, $|C \cap \omega|$ is positive and even. Let $D \subset \Omega$ be the 2-disk with $\partial D = C$. Note that $\sigma \cap \text{Int}\, D = \emptyset$. Also, $B := D \cap \omega$ is a nonempty, finite disjoint union of neatly embedded arcs in D. Recall that the only component of σ that meets τ is K, and K is essential in Ω. So, D is contained in the interior of a sector $S_{i,j}$ and $f \mid D$ is an embedding. Without loss of generality, assume $D \subset \text{Int}\, S_{2,3}$. Let $D' = f(D)$, an embedded 2-disk with $\partial D' = C' = f(C)$. The circle C' also bounds two 2-disks, D_1 and D_2, in Σ. As $\sigma \cap \text{Int}\, D = \emptyset$, $D' \cup D_1$ and $D' \cup D_2$ are embedded 2-spheres in $S^2[1,2]$. By Lemma 4.1 (using the arc T_1, say), one of these spheres is essential in $S^2[1,2]$ and the other is inessential. Without loss of generality, assume $D' \cup D_1$ is essential in $S^2[1,2]$ and $D' \cup D_2$ is inessential. Let Δ be the 3-disk in $S^2[1,2]$ with $\partial \Delta = D' \cup D_2$.

Lemma 6.7. *The disk D cannot be disjoint from the special points p^1 and p^2.*

Proof. Otherwise, let b be a component of B. Let $a \subset C$ be an arc with $\partial a = \partial b$. The arcs a and b contradict Lemma 6.6. $\qquad\square$

Lemma 6.8. *The disk D cannot contain both special points p^1 and p^2.*

Proof. Suppose otherwise. Note that T_1 meets D' twice (transversely in $\text{Int}\, D'$), and T_2 and T_3 are disjoint from D'. By Lemma 4.1, T is disjoint from D_2. The disk Δ permits construction of an ambient isotopy F_t, $0 \leq t \leq 1$, of $S^2[1,2]$ that carries D_2 past D' to a parallel copy of D'. This isotopy has support near Δ, is relative to a neighborhood of $T_2 \cup T_3 \cup \partial S^2[1,2]$, but does *not* fix T_1 setwise. Note that $F_1(T_1) \cap S'_{2,3} = \emptyset$ where $S'_{2,3}$ still denotes $f(S_{2,3})$. As $F_1(A) \approx A$ is a Borromean block, there is an ambient isotopy G_t, $0 \leq t \leq 1$, of $S^2[1,2]$, relative to a neighborhood of $\partial S^2[1,2]$, that straightens both $F_1(T_1)$ and $F_1(T_2) = T_2$. The strip $G_1(S'_{2,3})$ permits construction of an ambient isotopy H_t, $0 \leq t \leq 1$, of $S^2[1,2]$ that carries $G_1(F_1(T_3)) = G_1(T_3)$ to an arc close to and winding around $G_1(F_1(T_2)) = G_1(T_2)$. This isotopy has support near $G_1(S'_{2,3})$, is relative to both $G_1(F_1(T_1))$ and $G_1(F_1(T_2)) = G_1(T_2)$, but otherwise is not relative to $\partial S^2[1,2]$. The resulting block $B = H_1 G_1 F_1(A)$ is diffeomorphic to ε_3 by untwisting tangle components 2 and 3. So, $A \approx B \approx \varepsilon_3$, which contradicts Corollary 2.6. $\qquad\square$

Lemma 6.9. *The disk D cannot contain exactly one of the special points p^1 or p^2.*

Proof. Assume, without loss of generality, that $\operatorname{Int} D$ contains p^1 but not p^2. Recall that $\sigma \cap \operatorname{Int} D = \emptyset$. If B is connected, then Lemma 6.5 applies to D, a contradiction. If B is disconnected, then B contains a component, b, that does not contain p^1. Let $a \subset C$ be the arc with $\partial a = \partial b$ and such that p^1 does not lie inside the simple closed curve $a \cup b$. The arcs a and b contradict Lemma 6.6. $\qquad \square$

Taking stock, σ contains no component inessential in Ω. So, $\sigma = K$ is a single, essential circle in Ω, and K meets each τ_i at one point (transversely). It remains to prove that $K \cap \mathbb{T} = \emptyset$. A *segment* will mean an arc $s \subset K$ for which $s \cap (\omega \cup \tau) = \partial s$. Each inline figure, such as ⊢•⊣↕, represents a segment of K in a sector; vertical lines represent adjacent components of τ, upper and lower horizontal lines represent arcs of ω, dots represent special points, and arrows indicate reflected cases of the entire figure. For example, ⊢•⊣↕ represents six cases (three choices of sector and a possible vertical reflection), and ⊢•⊣↔ represents twelve cases.

Lemma 6.10. *None of the following appear*: ⊟•⊟, ⊢•⊣↕, *or* ⊢•⊣↔.

Proof. Observe that points of Ω inside (outside) K map under f to lie inside (outside) Σ respectively. Suppose there is ⊟•⊟. Without loss of generality, assume the indicated special points are p^1 and p^2. Then, $f(p^1)$ is outside Σ and $f(p^2)$ is inside Σ. There are three possibilities for the location of $K \cap \tau_1$ (see Fig. 17 at right). The inner option implies $f(p_1)$ and $f(p_2)$ both lie outside Σ. The middle option implies $f(p_1)$ lies inside Σ and $f(p_2)$ lies outside Σ. The outer option implies $f(p_1)$ and $f(p_2)$ both lie inside Σ. All three are contradictions since $f(p_1) = f(p^1)$ and $f(p_2) = f(p^2)$. So, no ⊟•⊟ appears.

We claim that:

There do not exist ⊢•⊣↕ and ⊢•⊣↕ in the same sector

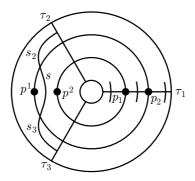and with the same reflection. $\qquad\qquad (6.1)$

Fig. 17. Hypothetical arrangement of segments of K.

To see this, suppose, by way of contradiction, that ⊞ and ⊞ both appear in the same sector, say $S_{2,3}$. Let z_+ and z_- denote these segments respectively. Assume, without loss of generality, that the special point just below z_+ is p^1. Let $z \subset \omega$ be the unique arc such that $p^1 \in \mathrm{Int}\, z$ and $z \cap \sigma = \partial z$. Lemma 6.6 implies that ∂z bounds segments s_+ parallel to z_+ and s_- parallel to z_- ($s_+ = z_+$ and $s_- = z_-$ are possible). Then, $s_+ \cup s_-$ is an inessential component of σ, a contradiction. This completes our proof of (6.1).

Suppose there is ⊞. Assume, without loss of generality, that this segment s of K lies in $S_{2,3}$. By Lemma 6.6, the only possible segment of K that meets both $\mathbb{T}$ and $\Omega - \mathbb{T}$, and is disjoint from τ, is ⊞$\updownarrow$. By (6.1), the boundary points of s lie in distinct segments, s_2 and s_3, of K, where s_2 meets τ_2 at one endpoint and s_3 meets τ_3 at one endpoint (see Fig. 17). So, $f(p^1)$ is outside Σ and $f(p^2)$ is inside Σ. This yields the same contradiction as for ⊞ above. Hence, no ⊞$\updownarrow$ appears.

Suppose there is ⊞. Call this segment s and assume, without loss of generality, that p^1 is the special point pictured above s. Let $z \subset \omega$ denote the short arc with $\partial z = \{q_2, p^1\}$ (see Fig. 18). We claim that no segment of K contained in $\mathbb{T}$ may meet z. This follows from: (i) Lemma 6.6, (ii) the nonexistence of ⊞$\updownarrow$, (iii) since s meets τ_2, and (iv) since K meets τ_2 exactly once. Therefore, $K \cap S_{2,3}$ appears as in Fig. 18. So, $f(p^1)$ is outside Σ and $f(p^2)$ is inside Σ, a familiar contradiction. Hence, no ⊞⤧ appears. □

Recall that $\mathbb{T}' \subset S^2[1,2]$ is an embedded torus, smooth except for corners along ω'. So, $\Sigma \cap \mathbb{T}'$ is a closed 1-manifold, smooth except for corners, and embedded in Σ and in $\mathbb{T}'$. We introduce a based longitude λ and a based meridian μ on $\mathbb{T}'$. Both originate and terminate at $f(p_2) \in \omega'$ (recall Figs. 2, 12, and 13). The longitude λ runs once along ω' in the CW direction about the z-axis when viewed from the point $(0,0,1)$. The meridian μ is the right loop in Fig. 13, oriented CCW, where
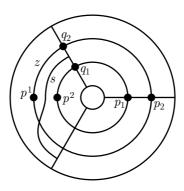

Fig. 18. K cannot exit the annulus $\mathbb{T}$ between q_2 and p^1.

the plane of intersection is the xz-plane. In particular, a parallel pushoff λ_0 of λ into the inside of $\mathbb{T}'$ has linking number $+1$ with μ in $\mathbb{R}^3$. An oriented loop in $\mathbb{T}'$ has *type* (m,n) provided it is freely homotopic in $\mathbb{T}'$ to $\mu^m \lambda^n$.

Lemma 6.11. *Let C be a component of $\Sigma \cap \mathbb{T}'$, equipped with an orientation and of type (m,n). Then, $mn = 0$.*

Proof. Assume C is essential in $\mathbb{T}'$ (otherwise, the result is clear). Note that $\gcd(m,n) = 1$. Focus attention on the submanifolds Σ and $\mathbb{T}'$ of $S^2[1,2] \subset \mathbb{R}^3$. We view Σ and $\mathbb{T}'$ as submanifolds of $S^3 = \mathbb{R}^3 \cup \{\infty\}$, where $S^3 = (S^1 \times D^2) \cup (D^2 \times S^1)$ and $(S^1 \times D^2) \cap (D^2 \times S^1) = \mathbb{T}'$. As C is essential in $\mathbb{T}'$, C is not null-homologous (denoted $C \not\sim 0$) in both $S^1 \times D^2$ and $D^2 \times S^1$. Let X denote $S^1 \times D^2$ or $D^2 \times S^1$ where $C \not\sim 0$ in X. Exactly one component of $\Sigma - \mathbb{T}'$ contains C in its frontier and lies in X; let Γ denote the closure of this component in Σ. Note that Γ is a compact 2-disk with holes, $C \subset \partial\Gamma$, $\Gamma \subset X$, and $\partial\Gamma \subset \partial X = \mathbb{T}'$. As $C \not\sim 0$ in X, there must be another component C_0 of $\partial\Gamma$ such that $C_0 \not\sim 0$ in X. In particular, C_0 is essential in $\mathbb{T}'$. Choose an orientation of C_0. Then, C_0 has type (m_0, n_0) where $\gcd(m_0, n_0) = 1$. The algebraic intersection number of C and C_0 in $\mathbb{T}'$ equals $mn_0 - nm_0$, which must vanish since $\Sigma \cap \mathbb{T}'$ is embedded in $\mathbb{T}'$. It follows that C_0 has type $\pm(m,n)$. Switching the orientation of C_0 if necessary, C_0 has type (m,n). Thus, C and C_0 are parallel in $\mathbb{T}'$. An exercise (left to the reader) shows that the linking number $\mathrm{lk}(C, C_0)$ in $\mathbb{R}^3$ equals mn. As C and C_0 are disjointly embedded in the sphere $\Sigma \subset \mathbb{R}^3$, $\mathrm{lk}(C, C_0) = 0$. Hence, $mn = 0$ as desired. $\square$

We define an *extended segment* to be an arc component of $K \cap \mathbb{T}$ of the form:

In the latter type, the vertical line is a component of τ and the horizontal lines are arcs of ω not meeting special points.

Corollary 6.12. *Any component of $K \cap \mathbb{T}$ is an extended segment.*

Proof. Let s be a component of $K \cap \mathbb{T}$. By Lemma 6.10, no appears. Therefore, $s \neq K$. Hence, s is an arc neatly embedded in $\mathbb{T}$. Note that $\partial s \subset \omega$, and ∂s is disjoint from τ and from special points. By Lemma 6.6, no appears. The result now follows from Lemma 6.10 by considering the possible locations of points in ∂s. $\square$

Proposition 6.13. $K \cap \mathbb{T} = \emptyset$.

Proof. Suppose, by way of contradiction, that K meets $\mathbb{T}$. Then, there is a circle $C \subset \Sigma \cap \mathbb{T}'$. By Corollary 6.12, there exists a finite, disjoint collection $s_1, s_2, \ldots, s_m$, $m \geq 1$, of extended segments such that:

$$C = s_1' \cup s_2' \cup \cdots \cup s_m'.$$

Orient all segments s_k, $1 \leq k \leq m$, to point out from $S^2[4/3]$. Note that this yields a coherent orientation of C. A moment of reflection on the immersion f (cf. Fig. 2) reveals that each s'_k winds around $\mathbb{T}'$ by: (i) $+1$ revolutions in the μ direction, and (ii) θ_k revolutions in the λ direction where $1/6 < \theta_k < 5/6$. Hence, C has type (m, n) where $m \geq 1$ and $m/6 < n < 5m/6$, which contradicts Lemma 6.11. $\qquad\square$

This completes our proof that A is irreducible and our proof of Theorem 6.1.

$\square$

7. Improving Spheres in Concatenations of Borromean Blocks

This section proves that certain spheres in concatenations of Borromean blocks may be ambiently isotoped into a single block summand, while fixing the tangle setwise.

Proposition 7.1. *Let B_i, $1 \leq i \leq k$, be Borromean blocks where $k \geq 2$. Consider the concatenation*:

$$B := B_1 B_2 \cdots B_k = (S^2[1, k+1], \tau).$$

Let Σ be a 2-sphere embedded in the interior of $S^2[1, k+1]$, transverse to τ, and meeting each component τ_i of τ at exactly one point x_i for $i = 1$, 2, and 3. Then, there is a neighborhood U of $\partial S^2[1, k+1]$ in $S^2[1, k+1]$ and an ambient isotopy H_t, $0 \leq t \leq 1$, of $S^2[1, k+1]$ such that:

$$H_0 = \mathrm{Id}. \tag{7.1}$$

$$H_t \,|\, U = \mathrm{Id} \text{ for all } 0 \leq t \leq 1. \tag{7.2}$$

$$H_t \text{ fixes } \tau \text{ setwise for all } 0 \leq t \leq 1. \tag{7.3}$$

$$H_1(\Sigma) \subset \mathrm{Int}\, S^2[m, m+1] \text{ for some } m \in \{1, 2, \ldots, k\}. \tag{7.4}$$

Remark 7.2. For the definition of *Borromean block*, see Sec. 2.3.

Proof. Without loss of generality, no point x_i lies in a sphere $S^2[j]$ where $j \in \{2, 3, \ldots, k\}$, and Σ is transverse to these spheres. So, $Z := \Sigma \cap (\bigcup_{j=2}^{k} S^2[j])$ is a closed 1-manifold. It suffices to improve Σ, by appropriate ambient isotopies of $S^2[1, k+1]$, so that Z becomes empty. We will employ the following two operations.

Operation 1. Suppose C is a component of Z bounding a disk $D_1 \subset \Sigma$ such that: (i) D_1 is disjoint from $Z - C$ and (ii) D_1 is disjoint from τ. The circle C lies in $S^2[j]$ for some $j \in \{2, 3, \ldots, k\}$ and bounds two 2-disks, D_2 and D'_2, in $S^2[j]$. By Lemma 5.3 (with $a = $ any τ_i), we have $0 = \#_2(a, D_1) = \#_2(a, D_2)$ and $D_1 \cup D_2$ is inessential in $S^2[1, k+1]$. So, $D_2 \cap \tau = \emptyset$ (since each τ_i meets D_2 at most once) and there is a 3-disk D in $S^2[1, k+1]$ bounded by $D_1 \cup D_2$. Hence, τ is disjoint from D and D permits construction of an ambient isotopy, with support near D, that carries D_1 past D_2 to a parallel copy of D_2. Thus, C (at least) has been eliminated from Z.

Operation 2. Suppose C is a component of Z bounding a disk $D_1 \subset \Sigma$ such that: (i) D_1 is disjoint from $Z - C$ and (ii) D_1 intersects τ at one point. For notational convenience, we assume $D_1 \cap \tau = \{x_1\}$. Note that $x_1 \in \text{Int } D_1$. The circle C lies in $S^2[j]$ for some $j \in \{2, 3, \ldots, k\}$. Let $D_2 \subset S^2[j]$ be the 2-disk whose boundary is C and whose interior meets τ_1 (necessarily at one point, call it q). Now, D_1 lies in $S^2[j-1, j]$ or in $S^2[j, j+1]$. Without loss of generality, assume the latter. Consider the block:

$$X := (S^2[j, j+1], r),$$

where r has components:

$$r_i := \tau_i \cap S^2[j, j+1] \quad \text{for } i = 1, 2, \text{ and } 3.$$

Let X' be the block obtained from X by forgetting r_3. Then, $X' \approx \varepsilon_2$ since $X \approx B_j$ is a Borromean block. Hence, Lemma 5.4 implies that $D_1 \cup D_2$ bounds the 3-disk $D \subset S^2[j, j+1]$, that $D \cap r_2 = \emptyset$, and there exists a diffeomorphism $g : (D, D \cap (r_1 \cup r_2)) \rightarrow (D_+^3, \{(0,0)\} \times D_+^1)$. The first two of these consequences imply that $D \cap \tau_2 = \emptyset$. Forgetting r_2 instead of r_3, we get $D \cap \tau_3 = \emptyset$. Therefore, we have a diffeomorphism:

$$g : (D, D \cap \tau) \rightarrow (D_+^3, \{(0,0)\} \times D_+^1).$$

This diffeomorphism permits construction of an ambient isotopy of $S^2[1, k+1]$ that: (i) has support near D, (ii) fixes τ_2 and τ_3 pointwise, (iii) fixes τ_1 setwise, and (iv) carries D_1 past D_2 to a parallel copy of D_2. Thus, C (at least) has been eliminated from Z.

Observe that if $Z \neq \emptyset$, then Operation 1 or 2 is applicable. Indeed, let C_0 be a component of Z. One component, W, of $\Sigma - C_0$ contains at most one of the points x_1, x_2, or x_3. If W contains components of Z, then Operation 1 or 2 applies to any innermost component of Z in W. If W contains no component of Z, then Operation 1 or 2 applies to C_0 itself. So, by finitely many applications of Operations 1 and 2, we get $Z = \emptyset$ and the proof is complete. $\qquad \square$

Corollary 7.3. *Let B_i, $1 \leq i \leq m$, and C_j, $1 \leq j \leq n$, be irreducible Borromean blocks. Consider the concatenations*:

$$B := B_1 B_2 \cdots B_m = (S^2[1, m+1], \tau),$$

$$C := C_1 C_2 \cdots C_n = (S^2[1, n+1], \sigma).$$

Suppose there is a diffeomorphism $f : B \rightarrow C$ such that $f(S^2) = S^2$. Then, $m = n$ and there is an isotopy of f, sending τ to σ setwise at all times, to a diffeomorphism $g : B \rightarrow C$ such that $g(B_i) = C_i$ for each $1 \leq i \leq m$.

Proof. Assume $m \leq n$ (otherwise, consider f^{-1}). If $m = 1$, then $n > 1$ contradicts irreducibility of B_1 (recall that each $C_j \not\approx \varepsilon_3$ since each C_j is a Borromean block). So, $n = 1$ and we are done. Next, let $m > 1$. All isotopies of f send τ to σ setwise

at all times. All isotopies of C are ambient and fix σ setwise at all times. By an abuse of notation, the corresponding improved f will still be denoted by f. By Proposition 7.1, we can isotop f such that $f(S^2[2]) \subset \operatorname{Int} C_j$ for some j. As C_j is irreducible, we can isotop f so that $f(S^2[2])$ is a level 2-sphere in C_j very close to one boundary sphere of C_j, namely: (i) $S^2[j]$, or (ii) $S^2[j+1]$.

Case 1: $j = 1$ and (i) occurs. Then, $B_1 \approx \varepsilon_3$, a contradiction since B_1 is a Borromean block.

Case 2: $j = 1$ and (ii) occurs, or $j = 2$ and (i) occurs. Then, we may further isotop f so that $f(S^2[2]) = S^2[2]$.

Case 3: $j = 2$ and (ii) occurs, or $j > 2$. These cases contradict irreducibility of B_1.

In any case, we have arranged that $f(S^2[2]) = S^2[2]$. Repeat this process with $f(S^2[3])$ and so forth, and it must terminate with $m = n$. $\qquad\square$

Corollary 7.4. *Let B_i and C_i, $i \in \mathbb{Z}^+$, be sequences of irreducible Borromean blocks. Consider the two Borromean rays*:

$$(\mathbb{R}^3, \tau) := D^3 B_1 B_2 B_3 \cdots,$$

$$(\mathbb{R}^3, \sigma) := D^3 C_1 C_2 C_3 \cdots.$$

Suppose there is a diffeomorphism $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$. Then, there exist $n \in \mathbb{Z}$ and $N \in \mathbb{Z}^+$, and there is an isotopy of f, sending τ to σ setwise at all times, to a diffeomorphism $g : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$ such that $g(B_i) = C_{i+n}$ for each $i \geq N$.

Proof. All isotopies send τ to σ setwise at all times, and are relative to D^3. By compactness, there exists an integer $N \geq 2$ such that $f(S^2[N])$ is disjoint from $D^3 C_1$. By Proposition 7.1, we may isotop f so that $f(S^2[N]) \subset \operatorname{Int} C_i$ for some $i \geq 2$. As C_i is irreducible, we may further isotop f so that $f(S^2[N]) = S^2[M]$ for some integer $M \geq 2$. Define $n := M - N \in \mathbb{Z}$. Use Proposition 7.1 and irreducibility repeatedly to get $f(S^2[i]) = S^2[i+n]$ for each integer $i \geq N$. $\qquad\square$

8. Borromean Rays and Hyperplanes

8.1. *Borromean rays*

This section proves Theorem 8.1, the first of our main results. If σ is a multiray given by a concatenation of blocks:

$$(\mathbb{R}^3, \sigma) = D^3 C_1 C_2 C_3 \cdots,$$

then the *mirror* of σ, denoted $\overline{\sigma}$, is defined by:

$$(\mathbb{R}^3, \overline{\sigma}) := D^3 \overline{C_1}\, \overline{C_2}\, \overline{C_3} \cdots.$$

Theorem 8.1. *Let B_i and C_i, $i \in \mathbb{Z}^+$, be sequences of blocks in $\mathcal{B}$. Consider the two Borromean rays*:

$$(\mathbb{R}^3, \tau) := D^3 B_1 B_2 B_3 \cdots,$$

$$(\mathbb{R}^3, \sigma) := D^3 C_1 C_2 C_3 \cdots.$$

There exists a diffeomorphism $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$ preserving orientation of $\mathbb{R}^3$ if and only if there exists $n \in \mathbb{Z}$ such that one of the following holds for all sufficiently large $i \in \mathbb{Z}^+$:

$$B_i = C_{i+n} \text{ (i.e. } B_i \text{ and } C_i \text{ have identical tails).} \tag{8.1}$$

$$B_i = \overline{C_{i+n}}^* \text{ (i.e. } B_i \text{ and } \overline{C_i}^* \text{ have identical tails).} \tag{8.2}$$

There exists a diffeomorphism $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$ reversing orientation of $\mathbb{R}^3$ if and only if there exists $n \in \mathbb{Z}$ such that one of the following holds for all sufficiently large $i \in \mathbb{Z}^+$:

$$B_i = \overline{C_{i+n}} \text{ (i.e. } B_i \text{ and } \overline{C_i} \text{ have identical tails).} \tag{8.3}$$

$$B_i = C_{i+n}{}^* \text{ (i.e. } B_i \text{ and } C_i{}^* \text{ have identical tails).} \tag{8.4}$$

Proof. If (8.1) holds, then f exists by Remarks 2.16(1). Assume (8.2) holds. Then:

$$D^3 B_1 B_2 B_3 \cdots \approx D^3 B_N B_{N+1} B_{N+2} \cdots$$

$$= D^3 \overline{C_M}^* \overline{C_{M+1}}^* \overline{C_{M+2}}^* \cdots$$

$$\approx D^3 \overline{C_1}^* \overline{C_2}^* \overline{C_3}^* \cdots$$

$$\approx D^3 C_1 C_2 C_3 \cdots.$$

The last diffeomorphism is $(x, y, z) \mapsto (x, -y, -z)$ (= rotation of $\mathbb{R}^3$ about x-axis), followed by a simple ambient isotopy in each block relative to boundary 2-spheres (cf. Sec. 3). The other two diffeomorphisms come from Remarks 2.16(1). The composition is the required f.

Assume (8.3) holds. The first orientation preserving case above yields a diffeomorphism $g : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \overline{\sigma})$ that preserves orientation of $\mathbb{R}^3$. Composing g with $(x, y, z) \mapsto (x, y, -z)$ yields the required f. Assume (8.4) holds. Then:

$$B_i = \overline{(\overline{C_{i+n}})}^*$$

for sufficiently large i. The second orientation preserving case yields a diffeomorphism $g : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \overline{\sigma})$ that preserves orientation of $\mathbb{R}^3$. Again, compose g with $(x, y, z) \mapsto (x, y, -z)$ to obtain the required f.

For the forward implications, note that blocks in $\mathcal{B}$ are irreducible by Theorem 6.1. By Corollary 7.4, there exist $n \in \mathbb{Z}$, $N \in \mathbb{Z}^+$, and a diffeomorphism $g : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \sigma)$, isotopic to f, such that $g(B_i) = C_{i+n}$ for each $i \geq N$. Assume

f preserves orientation of $\mathbb{R}^3$. Then, g preserves orientation of $\mathbb{R}^3$. So, each of the diffeomorphisms:

$$g| : B_i \to C_{i+n}, \quad i \geq N,$$

preserves orientation, preserves boundary 2-spheres setwise, and has the same tangle permutation $\pi \in \mathrm{Sym}(3)$ (see Sec. 3). If $\pi \in A_3$, then (8.1) holds by Table 2 in Sec. 3. If $\pi \in (1,2)A_3 = C$, then (8.2) holds by Table 2.

Finally, assume the given f reverses orientation of $\mathbb{R}^3$. The composition of f with $(x, y, z) \mapsto (x, y, -z)$ is a diffeomorphism $(\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \overline{\sigma})$ preserving orientation of $\mathbb{R}^3$. Now, apply the orientation preserving case. □

Let $\mathcal{S}$ denote the set of all sequences B_i, $i \in \mathbb{Z}^+$, of blocks in $\mathcal{B}$. Declare two sequences to be equivalent, written $B_i \sim C_i$, if and only if their corresponding Borromean rays are equivalent by some diffeomorphism of $\mathbb{R}^3$ (not necessarily orientation preserving).

Corollary 8.2. *The set of equivalence classes $\mathcal{S}/\!\sim$ is uncountable.*

Proof. By Theorem 8.1, the equivalence class of any given sequence is countable. □

A multiray $\tau \subset \mathbb{R}^3$ is *achiral* provided there exists a diffeomorphism $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \overline{\tau})$ preserving orientation of $\mathbb{R}^3$. Otherwise, τ is *chiral*. Equivalently, τ is *achiral* provided there exists a diffeomorphism $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \tau)$ reversing orientation of $\mathbb{R}^3$.

Corollary 8.3. *Let $(\mathbb{R}^3, \tau) = D^3 B_1 B_2 B_3 \cdots$ for a sequence B_i, $i \in \mathbb{Z}^+$, of blocks in $\mathcal{B}$. Then, τ is achiral if and only if* (i) *there exists a block $C = C_1 C_2 \cdots C_k$ where $k \in \mathbb{Z}^+$ and each $C_i \in \mathcal{B}$, and* (ii) *a tail of the sequence B_i equals one of the following*:

$$C\overline{C}C\overline{C}C\overline{C}\cdots \quad or \quad CC^*CC^*CC^*\cdots. \tag{8.5}$$

In particular, if τ is achiral, then B_i is eventually periodic. So, at most countably many achiral τ arise this way.

Proof. By Theorem 8.1, τ is achiral if and only if (i) B_i and $\overline{B_i}$ have identical tails, or (ii) B_i and B_i^* have identical tails. So, if a tail of B_i has the form (8.5), then τ is achiral. Conversely, suppose B_i and $\overline{B_i}$ have identical tails (the other case is similar). Then, there exists $n \in \mathbb{Z}$ and $N \in \mathbb{Z}^+$ such that

$$B_i = \overline{B_{i+n}} \quad \text{for all } i \geq N. \tag{8.6}$$

As $B_N \neq \overline{B_N}$, we get $n \neq 0$. Without loss of generality, assume $n > 0$ (otherwise, apply bar to (8.6)). Note that:

$$B_{N+n} = \overline{\overline{B_{N+n}}} = \overline{B_N},$$

where the second equality used (8.6). Repeating this argument, we get that if $m \geq 0$ and $0 \leq j < n$, then:

$$B_{N+mn+j} = \begin{cases} B_{N+j} & \text{if } m \text{ is even,} \\ \overline{B_{N+j}} & \text{if } m \text{ is odd.} \end{cases}$$

Therefore, $C = B_N B_{N+1} \cdots B_{N+n-1}$. $\qquad\square$

Example 8.4. For each $m \in \mathbb{Z}^+$, define:

$$C_m := A^m = \underbrace{AA \cdots A}_{m \text{ times}}$$

and define:

$$(\mathbb{R}^3, \tau_m) := D^3 C_m \overline{C_m} C_m \overline{C_m} C_m \overline{C_m} \cdots.$$

By Corollary 8.3, τ_m is achiral. By Theorem 8.1, m is a diffeomorphism invariant of τ_m. So, τ_m, $m \in \mathbb{Z}^+$, is a countably infinite family of achiral Borromean rays, pairwise distinct up to diffeomorphism.

8.2. *Borromean hyperplanes*

A *hyperplane* is a smooth, proper embedding of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$. A *multiple hyperplane* $H \subset \mathbb{R}^n$ is a smooth, proper embedding of a disjoint union of (at most countably many) copies of $\mathbb{R}^{n-1}$. The basic invariant of H is an associated tree $T(H)$. The vertices of $T(H)$ are the components of $\mathbb{R}^n - H$. Two vertices are adjacent provided their closures in $\mathbb{R}^n$ share a component of H. Figure 19 depicts some multiple hyperplanes and their trees.

Multiple hyperplanes in $\mathbb{R}^n$, $n \neq 3$, are classified by their associated trees [3, Sec. 9]. More precisely, if H and H' are multiple hyperplanes in $\mathbb{R}^n$, $n > 3$, and the trees $T(H)$ and $T(H')$ are isomorphic, then there is a diffeomorphism $f : (\mathbb{R}^n, H) \to (\mathbb{R}^n, H')$ that preserves orientation of $\mathbb{R}^n$. For $n = 2$, these trees are naturally planar (i.e. the edges incident with a given vertex are cyclically ordered). The result then holds provided $T(H)$ and $T(H')$ are isomorphic as planar trees.

Let $\mathbb{H}^3$ denote Klein's model of hyperbolic 3-space. Namely, $\mathbb{H}^3$ is the open unit 3-disk in $\mathbb{R}^3$, and a *hyperbolic hyperplane* is the nonempty intersection of $\mathbb{H}^3$
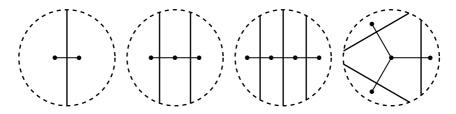

Fig. 19.   Multiple hyperplanes and their associated trees.

with an affine plane in $\mathbb{R}^3$. A *hyperbolic multiple hyperplane* is a properly embedded submanifold of $\mathbb{H}^3$, each component of which is a hyperbolic hyperplane. The multiple hyperplanes in Fig. 19 are hyperbolic. A multiple hyperplane $H \subset \mathbb{R}^3$ is *unknotted* provided there exists a diffeomorphism $f : (\mathbb{R}^3, H) \to (\mathbb{H}^3, L)$ where L is a hyperbolic multiple hyperplane.

Lemma 8.5. *Let H and H' be unknotted multiple hyperplanes in $\mathbb{R}^3$. There exists a diffeomorphism $f : (\mathbb{R}^3, H) \to (\mathbb{R}^3, H')$ preserving orientation of $\mathbb{R}^3$ if and only if $T(H)$ and $T(H')$ are isomorphic trees.*

Proof. The classification proof for multiple hyperplanes in $\mathbb{R}^n$, $n > 3$, applies to *unknotted* multiple hyperplanes in $\mathbb{R}^3$ [3, Sec. 9]. $\qquad\square$

Given a tree T that is at most countable and is not necessarily locally finite, it is not difficult to construct a hyperbolic multiple hyperplane $L \subset \mathbb{H}^3$ such that $T(L)$ is isomorphic to T. Hence, unknotted multiple hyperplanes in $\mathbb{R}^3$ are classified, up to diffeomorphism, by isomorphism classes of such trees. Up to isomorphism, there is a unique tree with n vertices for $n \leq 3$. For $n = 4$, there are two: a linear tree and the 3-prong (see Fig. 19).

Let $\mathbb{R}^3_+ := \{(x, y, z) \,|\, z \geq 0\}$ denote closed upper half-space. Throughout this section, $\zeta \subset \mathbb{R}^3$ denotes the ray $\{(0,0)\} \times [1, \infty)$ in the positive z-axis.

If $r \subset \mathbb{R}^3$ is a multiray with $1 \leq n \leq \infty$ components, then νr denotes a smooth, closed regular neighborhood of r in $\mathbb{R}^3$ [3, Sec. 3] (see also [13]). If r_i is a component of r, then νr_i denotes the component of νr containing r_i. Basic properties of νr include:

For each r_i, there is a diffeomorphism of pairs $\varphi_i : (\nu r_i, r_i) \to (\mathbb{R}^3_+, \zeta)$. $\qquad$ (8.7)

νr is unique up to ambient isotopy of $\mathbb{R}^3$ relative to r. $\qquad$ (8.8)

The boundary of νr, denoted $\partial \nu r$, is an n-component multiple

$\quad$ hyperplane. $\qquad$ (8.9)

The tree $T(\partial \nu r)$ is an n-prong. $\qquad$ (8.10)

We wish to show that each multiray $r \subset \mathbb{R}^3$ is essentially determined by the multiple hyperplane $H = \partial \nu r$.

Lemma 8.6. *Let $\alpha : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ be a diffeomorphism. Then, α is isotopic, relative to a neighborhood of $\partial \mathbb{R}^3_+$ in $\mathbb{R}^3_+$, to a diffeomorphism of pairs $\beta : (\mathbb{R}^3_+, \zeta) \to (\mathbb{R}^3_+, \zeta)$.*

Proof. Let $\tau \subset \mathbb{R}^3$ be the ray $[1, \infty) \times \{(0,0)\}$ in the positive x-axis. Let S denote the points of norm ≥ 1 in the first quadrant of the xz-plane. We identify S with $[0,1] \times [1, \infty)$ so that $\tau = \{0\} \times [1, \infty)$ and $\zeta = \{1\} \times [1, \infty)$. All isotopies of $\mathbb{R}^3_+$ will be ambient and relative to a neighborhood of $\partial \mathbb{R}^3_+$ in $\mathbb{R}^3_+$. Isotoped subsets will be called by their original names. It suffices to construct an isotopy of $\mathbb{R}^3_+$ that carries

$\alpha(\zeta)$ to ζ. Let $\pi : \mathbb{R}^3_+ \to \mathbb{R}^2 \times \{0\}$ be orthogonal projection. The submanifold $\alpha(S)$ permits construction of an isotopy of $\mathbb{R}^3_+$ that carries $\alpha(\zeta)$ close enough to $\alpha(\tau)$ so that $\pi \,|\, \alpha(\zeta)$ is an embedding. This is possible since $\alpha(\tau)$ is properly embedded. Next, by integrating a suitable vector field of the form $v(x, y, z) = (0, 0, *)$, we get an isotopy carrying $\alpha(\zeta)$ into $\mathbb{R}^2 \times \{1\}$. Rays do not knot in $\mathbb{R}^2$ [3, Theorem 9.13]. So, there is an ambient isotopy F of $\mathbb{R}^2 \times \{1\}$ that carries $\alpha(\zeta)$ to a straight ray. Use a small tube about $\mathbb{R}^2 \times \{1\}$ in $\mathrm{Int}\,\mathbb{R}^3_+$ and a suitable bump function to extend F to an isotopy of $\mathbb{R}^3_+$. The rest is elementary. $\qquad\square$

Remarks 8.7. (1) Lemma 8.6 is very different from "uniqueness of regular neighborhoods". Let K be a smooth subcomplex of a smooth manifold M. It is not true, in general, that each orientation preserving diffeomorphism $h : \nu K \to \nu K$ is isotopic to a diffeomorphism $(\nu K, K) \to (\nu K, K)$. For a simple counterexample, let K be a bouquet of three circles embedded in $M = \mathbb{R}^2$ in such a way that no circle of K is inside another. Then, $\nu K \subset M$ is a smooth, compact 2-disk with three holes. Let $C \subset \mathrm{Int}\,\nu K$ be a simple closed curve such that two boundary components of νK are inside C. Let $h : \nu K \to \nu K$ be a Dehn twist about C. Then, h is not isotopic to a diffeomorphism $(\nu K, K) \to (\nu K, K)$. Otherwise, h would be isotopic to the identity, which is false [9, pp. 239–247].

(2) Counterexamples exist even when K is a smooth submanifold. We are indebted to Bob Gompf for these examples. Let X be a simply connected, closed, symplectic 4-manifold with positive signature, denoted $\sigma(X) > 0$, and $b_2^+ > 1$. (Many such manifolds are known—even Kähler examples have been around for several decades.) As X is symplectic, X is smooth and oriented, and b_2^+ is odd. Let K be X blown up $\sigma(X)$ times. Then, $\sigma(K) = 0$ and K is homeomorphic to $Z := \sharp_m(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2)$ for some $m > 1$ (and very large in practice). Since K is symplectic, K has nonvanishing Seiberg–Witten invariants, denoted $\mathrm{SW}(K) \neq 0$. However, $\overline{K}$ ($= K$ with reversed orientation) splits off a $\mathbb{C}P^2$ summand (from the blowup of X) and has $b_2^+ > 1$. Therefore, $\mathrm{SW}(\overline{K}) = 0$ and K admits no orientation reversing self-diffeomorphism. Fix any $n \geq 2$. Then, $M := K \times S^n$ is a smooth, closed, oriented manifold containing $K = K \times \{p\}$, and $\nu K \subset M$ is identified with $K \times D^n$. Now, there is a smooth h-cobordism W between K and Z. So, $W \times S^{n-1}$ is smoothly a product, and, working relative to boundary, $W \times D^n$ is smoothly a product. Hence, there is a diffeomorphism $K \times D^n \to Z \times D^n$. Let α be an orientation reversing self-diffeomorphism of Z (e.g. α permutes the summands and is otherwise the identity). Let β be an orientation reversing self-diffeomorphism of D^n. Then, $k := \alpha \times \beta$ is an orientation preserving self-diffeomorphism of $Z \times D^n$, and k_* is multiplication by -1 on $H_4(Z \times D^n; \mathbb{Z})$. Let h be the corresponding orientation preserving self-diffeomorphism of νK. On $H_4(\nu K; \mathbb{Z})$, h_* is multiplication by -1. So, h is not isotopic to a diffeomorphism $(\nu K, K) \to (\nu K, K)$. Otherwise, we get a forbidden orientation reversing self-diffeomorphism of K.

(3) An alternative approach to proving Lemma 8.6 uses the following lemma together with some collaring arguments.

Lemma 8.8. *Let* $\gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *be an orientation preserving diffeomorphism. Then,* γ *is isotopic to the identity.*

Proof. As $\gamma(\mathbb{R}^{n-1} \times \{0\}) = \mathbb{R}^{n-1} \times \{0\}$, the well-known proof for a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ applies (see [19, p. 34]). $\square$

The following lemma is useful.

Lemma 8.9. *Let* $H \subset \mathbb{R}^n$ *be a hyperplane. Let* X *and* Y *denote the closures in* $\mathbb{R}^n$ *of the components of* $\mathbb{R}^n - H$. *The following are equivalent*:

$$\text{There is an ambient isotopy } F \text{ of } \mathbb{R}^n \text{ such that } F_1(H) = \mathbb{R}^{n-1} \times \{0\}. \quad (8.11)$$

$$\text{There is a diffeomorphism } f : (\mathbb{R}^n, H) \to (\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\}). \quad (8.12)$$

$$\text{There are diffeomorphisms } \varphi : X \to \mathbb{R}^n_+ \text{ and } \psi : Y \to \mathbb{R}^n_+. \quad (8.13)$$

Proof. Only two implications require proof.

$(8.12) \Rightarrow (8.11)$: We may assume f preserves orientation of $\mathbb{R}^n$ (otherwise, compose f with reflection through $\mathbb{R}^{n-1} \times \{0\}$). By Milnor [19, p. 34], f is isotopic to the identity.

$(8.13) \Rightarrow (8.12)$: Let $\mathbb{R}^n_-$ denote closed lower half-space. Replacing φ and ψ with their compositions with appropriate reflections, we can and do assume $\varphi : X \to \mathbb{R}^n_+$ and $\psi : Y \to \mathbb{R}^n_-$ are orientation preserving diffeomorphisms. Let $\mu = \psi \circ \varphi^{-1} \, | \, \mathbb{R}^{n-1} \times \{0\}$, an orientation preserving automorphism of $\mathbb{R}^{n-1} \times \{0\}$. Let η be the orientation preserving automorphism of $\mathbb{R}^{n-1}$ given by the following composition where pr is the obvious projection:

$$\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^{n-1} \times \{0\} \xrightarrow{\mu} \mathbb{R}^{n-1} \times \{0\} \xrightarrow{\text{pr}} \mathbb{R}^{n-1}.$$

Let Id be the identity map on $[0, \infty)$. Then, $\eta \times \text{Id}$ is an orientation preserving automorphism of $\mathbb{R}^n_+$. Replacing φ with $(\eta \times \text{Id}) \circ \varphi$, we can and do further assume $\varphi \, | \, H = \psi \, | \, H$. Define $h : \mathbb{R}^n \to \mathbb{R}^n$ by $h(p) = \varphi(p)$ if $p \in X$ and $h(p) = \psi(p)$ if $p \in Y$. Then, h is an orientation preserving autohomeomorphism of $\mathbb{R}^n$. By construction, h is smooth on X and h is smooth on Y. Using collaring uniqueness [14, Theorem 8.1.9], we may adjust h (by isotoping φ and ψ near H and relative to H) to obtain the desired diffeomorphism f. $\square$

Corollary 8.10. *Let* r *and* r' *be multirays in* $\mathbb{R}^3$. *Let* $h : \nu r \to \nu r'$ *be a diffeomorphism. Let* $H = \partial \nu r$ *and* $H' = \partial \nu r'$. *Then,* h *is isotopic, relative to a neighborhood of* H *in* νr, *to a diffeomorphism of pairs* $g : (\nu r, r) \to (\nu r', r')$.

Proof. It suffices to consider $h \, | \, \nu r_i$ where r_i is a component of r. Reindex the components of r' so that $h(\nu r_i) = \nu r'_i$. Let $\varphi_i : (\nu r_i, r_i) \to (\mathbb{R}^3_+, \zeta)$ and $\psi_i :$

$(\nu r_i', r_i') \to (\mathbb{R}^3_+, \zeta)$ be diffeomorphisms. Lemma 8.6 yields an isotopy $F : \mathbb{R}^3_+ \times [0,1] \to \mathbb{R}^3_+$, relative to a neighborhood of $\partial \mathbb{R}^3_+$, such that $F_0 = \psi_i h \varphi_i^{-1}$ and $F_1(\zeta) = \zeta$. Then, $\psi_i^{-1} \circ F \circ (\varphi_i \times \mathrm{Id})$ is the desired isotopy of $h \mid \nu r_i$. $\qquad \square$

Corollary 8.11. *If $r \subset \mathbb{R}^3$ is a knotted multiray, then $H := \partial \nu r$ is a knotted multiple hyperplane.*

Proof. Suppose H is unknotted. Then, there is a diffeomorphism $h : (\mathbb{R}^3, H) \to (\mathbb{H}^3, L)$ where L is a hyperbolic multiple hyperplane. As $T(H)$ is an n-prong, so is $T(L)$. Without loss of generality, the origin of $\mathbb{H}^3$ does not lie in $h(\nu r)$. For each component r_i of r, let $C_i = h(\nu r_i)$ and let $L_i = \partial C_i$. Let p_i be the point of L_i closest to the origin in $\mathbb{H}^3$ (for the euclidean metric). Let $\sigma_i \subset C_i$ be the radial ray in $\mathbb{H}^3$ with initial point p_i. Let $\tau_i \subset \mathrm{Int}\, C_i$ be the radial ray in σ_i that is half as long as σ_i (for the euclidean metric). Notice that C_i is a smooth, closed regular neighborhood of τ_i in $\mathbb{H}^3$. Let $\tau \subset \mathbb{H}^3$ be the radial multiray with components τ_i. Notice that $h(\nu r)$ is a smooth, closed regular neighborhood of τ in $\mathbb{H}^3$. As in Corollary 8.10, we may isotop h to a diffeomorphism $(\mathbb{R}^3, r) \to (\mathbb{H}^3, \tau)$. But, this implies r is unknotted, a contradiction. $\qquad \square$

Proposition 8.12. *Let τ and τ' be multirays in $\mathbb{R}^3$, each containing n components where $1 \leq n \leq \infty$. Let $H = \partial \nu \tau$ and let $H' = \partial \nu \tau'$. If $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \tau')$ is a diffeomorphism, then f is isotopic relative to τ to a diffeomorphism $g : (\mathbb{R}^3, H) \to (\mathbb{R}^3, H')$. Conversely, suppose $g : (\mathbb{R}^3, H) \to (\mathbb{R}^3, H')$ is a diffeomorphism. If any of the following conditions are met, then g is isotopic relative to $G := \mathbb{R}^3 - \mathrm{Int}\, \nu \tau$ to a diffeomorphism $f : (\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \tau')$.*

$$g(\nu \tau) = \nu \tau'. \tag{8.14}$$

$$n \geq 2. \tag{8.15}$$

$$n = 1, \text{ and } \tau \text{ or } \tau' \text{ is knotted.} \tag{8.16}$$

Remark 8.13. The case $n = 1$ where τ and τ' are unknotted is exceptional for trivial reasons. For example, let τ and τ' both equal ζ. Let $\nu \tau$ and $\nu \tau'$ both equal $\mathbb{R}^3_+$, so $H = H' = \mathbb{R}^2 \times \{0\}$. Then, $g(x, y, z) = (-x, y, -z)$ is an orientation preserving diffeomorphism of $\mathbb{R}^3$ sending H to H'. However, g is not isotopic relative to H to a diffeomorphism $(\mathbb{R}^3, \tau) \to (\mathbb{R}^3, \tau')$.

Proof of Proposition 8.12. The forward implication is immediate by ambient uniqueness of closed regular neighborhoods [3, Sec. 3]. Next, let $g : (\mathbb{R}^3, H) \to (\mathbb{R}^3, H')$ be a diffeomorphism. First, assume condition (8.14). Then, the result is immediate by Corollary 8.10. Second, assume condition (8.15). As $T(H)$ and $T(H')$ are both n-prongs, condition (8.14) is satisfied and the result follows. Third, assume condition (8.16). Without loss of generality, assume τ is knotted (otherwise, consider g^{-1}). Let $X = \nu \tau$ and let $Y = \mathbb{R}^3 - \mathrm{Int}\, \nu \tau$. By Corollary 8.11, H is knotted. By

Lemma 8.9, $Y \not\approx \mathbb{R}^3_+$. As $\nu\tau' \approx \mathbb{R}^3_+$, we see that condition (8.14) is satisfied and the result follows. □

Proposition 8.12 permits us to translate results on knotted multirays in $\mathbb{R}^3$ to results on knotted multiple hyperplanes in $\mathbb{R}^3$. By McPherson [18], there exist uncountably many knot types of a ray: so there exist uncountably many knot types of a hyperplane. By Fox and Harrold [11], there exist uncountably many knot types of two-component multirays with unknotted components (see [10, p. 988] for a nice example): so there exist uncountably many knot types of two-component multiple hyperplanes with unknotted components.

Proposition 8.12 was proved via ambient isotopies, so it also yields results on chirality. Let $H \subset \mathbb{R}^3$ be a multiple hyperplane. We say H is *achiral* provided there is a diffeomorphism $(\mathbb{R}^3, H) \to (\mathbb{R}^3, H)$ that reverses orientation of $\mathbb{R}^3$. Otherwise, H is *chiral*. For example, it is an exercise to show that each unknotted multiple hyperplane $H \subset \mathbb{R}^3$ is achiral.

A multiple hyperplane H in $\mathbb{R}^3$ forms *Borromean hyperplanes* provided H is knotted, but any two components of H form an unknotted multiple hyperplane. Proposition 8.12 implies that if $\tau \subset \mathbb{R}^3$ forms Borromean rays, then $\partial\nu r$ forms Borromean hyperplanes. Thus, we obtain our second main result.

Theorem 8.14. *There exist uncountably many Borromean hyperplanes, pairwise distinct up to diffeomorphism of $\mathbb{R}^3$. There exists a countably infinite family of achiral Borromean hyperplanes, pairwise distinct up to diffeomorphism of $\mathbb{R}^3$.*

Proof. Immediate by Proposition 8.12, Corollary 8.2, and Example 8.4. □

References

[1] W. R. Alford and B. J. Ball, Some almost polyhedral wild arcs, *Duke Math. J.* **30** (1963) 33–38.

[2] G. Burde and H. Zieschang, *Knots* (Walter de Gruyter & Co., Berlin, 2003).

[3] J. S. Calcut, H. C. King and L. C. Siebenmann, Connected sum at infinity and Cantrell–Stallings hyperplane unknotting, *Rocky Mountain J. Math.* **42** (2012) 1803–1862.

[4] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* $(\Gamma_4 = 0)$, Lecture Notes in Mathematics, Vol. 53 (Springer, Berlin, 1968).

[5] H. Debrunner and R. Fox, A mildly wild imbedding of an n-frame, *Duke Math. J.* **27** (1960) 425–429.

[6] P. H. Doyle, A wild triod in three-space, *Duke Math. J.* **26** (1959) 263–267.

[7] E. Fadell, Homotopy groups of configuration spaces and the string problem of Dirac, *Duke Math. J.* **29** (1962) 231–242.

[8] E. Fadell and J. Van Buskirk, The braid groups of E^2 and S^2, *Duke Math. J.* **29** (1962) 243–257.

[9] B. Farb and D. Margalit, *A Primer on Mapping Class Groups* (Princeton University Press, Princeton, 2012).

[10] R. H. Fox and E. Artin, Some wild cells and spheres in three-dimensional space, *Ann. of Math.* (2) **49** (1948) 979–990.

[11] R. H. Fox and O. G. Harrold, The Wilder arcs, in *Topology of 3-Manifolds and Related Topics* (Prentice-Hall, Englewood Cliffs, NJ, 1962), pp. 184–187.

[12] A. Hatcher, Notes on basic 3-manifold topology, preprint (2000), http://www.math.cornell.edu/~hatcher/3M/3M.pdf.

[13] M. W. Hirsch, Smooth regular neighborhoods, *Ann. of Math.* (2) **76** (1962) 524–530.

[14] M. W. Hirsch, *Differential Topology* (Springer, New York, 1994) (Corrected reprint of the 1976 original).

[15] S. Kinoshita, Alexander polynomials as isotopy invariants. II, *Osaka Math. J.* **11** (1959) 91–94.

[16] A. A. Kosinski, *Differential Manifolds*, with a new appendix by J. W. Morgan (Dover, Mineola, 2007) (Corrected reprint of the 1993 original).

[17] C. Livingston, Knotted symmetric graphs, *Proc. Amer. Math. Soc.* **123** (1995) 963–967.

[18] J. M. McPherson, Wild arcs in three-space. I. Families of Fox–Artin arcs, *Pacific J. Math.* **45** (1973) 585–598.

[19] J. Milnor, *Topology from the Differentiable Viewpoint*, Princeton Landmarks in Mathematics (Princeton University Press, Princeton, NJ, 1997).

[20] E. E. Moise, *Geometric Topology in Dimensions* 2 *and* 3 (Springer, New York, 1977).

[21] J. Munkres, Differentiable isotopies on the 2-sphere, *Michigan Math. J.* **7** (1960) 193–197.

[22] K. Murasugi and B. I. Kurpita, *A Study of Braids* (Kluwer Academic Publishers, Dordrecht, 1999).

[23] M. H. A. Newman, On a string problem of Dirac, *J. London Math. Soc.* **17** (1942) 173–177.

[24] D. Rolfsen, *Knots and Links* (Publish or Perish, Houston, 1990) (Corrected reprint of the 1976 original).

[25] T. B. Rushing, *Topological Embeddings*, Pure and Applied Mathematics, Vol. 52 (Academic Press, New York, 1973).

[26] J. K. Simon and K. Wolcott, Minimally knotted graphs in S^3, *Topology Appl.* **37** (1990) 163–180.

[27] S. Smale, Diffeomorphisms of the 2-sphere, *Proc. Amer. Math. Soc.* **10** (1959) 621–626.

[28] J. Stillwell, *Classical Topology and Combinatorial Group Theory* (Springer, New York, 1980).

[29] W. P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, Princeton, NJ, 1997).

[30] J. H. C. Whitehead, On the asphericity of regions in a 3-sphere, *Fund. Math.* **32** (1939) 149–166.