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# CALCULATING THE FUNDAMENTAL GROUP OF AN ORBIT SPACE

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ABSTRACT. Suppose $G$ acts effectively as a group of homeomorphisms of the connected, locally path connected, simply connected, locally compact metric space $X$. Let $\overline{G}$ denote the closure of $G$ in Homeo($X$), and $N$ the smallest normal subgroup of $\overline{G}$ which contains the path component of the identity of $\overline{G}$ and all those elements of $\overline{G}$ which have fixed points. We show that $\pi_1(X/G)$ is isomorphic to $\overline{G}/N$ subject to a weak path lifting assumption for the projection $X \to X/\overline{G}$.

Given a topological space $X$ together with a group $G$ of homeomorphisms of $X$, what can we say about the fundamental group of the orbit space $X/G$? Results for simplicial and discontinuous groups have been given in [**1**] and [**2**]. The object of this note is to produce a theorem which can deal with both discontinuous and continuous actions.

We shall assume that $X$ is a connected, locally path connected, locally compact metric space. Let $G$ be a group of homeomorphisms of $X$ which acts effectively on $X$, so that we can think of $G$ as a subgroup of the group Homeo($X$) of all homeomorphisms of $X$ endowed with the compact open topology.

Under very reasonable hypotheses (see conditions A, B, C below) the answer to our question is as follows. Let $\overline{G}$ denote the closure of $G$ in Homeo($X$), and let $N$ be the smallest normal subgroup of $\overline{G}$ which contains the path component of the identity of $\overline{G}$ and all those elements of $\overline{G}$ which have fixed points. Then if $X$ is simply connected the fundamental group of $X/G$ is isomorphic to the quotient group $\overline{G}/N$.

Suppose $X$ fails to be simply connected but has a universal covering space $\tilde{X}$. Each homeomorphism $g\colon X \to X$ lifts to a homeomorphism of $\tilde{X}$, and any two lifts of the same $g$ differ by a covering transformation. Therefore we have an action of an extension of $\pi_1(X)$ by $G$ on $\tilde{X}$ whose orbit space is homeomorphic to $X/G$, and we can apply our result in this setting. Details of the group extension and of its action on $\tilde{X}$ can be found in [**5**] and [**3**].

Here are some examples to illustrate a variety of situations in which the result can be used.

EXAMPLE 1. Take $S^2 \times \mathbf{R}$ for $X$ and $S^1 \times \mathbf{Z}$ for $G$, the action being as follows. The circle acts on $S^2$ by rotation leaving the north and south poles fixed, and acts trivially on $\mathbf{R}$. The generator of $\mathbf{Z}$ reflects $S^2$ in the equator and translates $\mathbf{R}$ along

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one unit. Then $G = \overline{G}$, $N = S^1 \times \{0\}$ and the fundamental group of the orbit space (which is easily seen to be the Möbius strip) is $\mathbf{Z}$.

EXAMPLE 2. Consider the action of $G = \mathrm{PSL}(2, \mathbf{Z})$ on the upper half plane $U$ as a group of linear fractional transformations $z \mapsto (az + b)/(cz + d)$. Then $G$ is a closed subgroup of Homeo($U$), and is generated by the elements $z \mapsto -\frac{1}{z}$, $z \mapsto 1 - \frac{1}{z}$ both of which have fixed points. Therefore the orbit space $U/G$ is simply connected. In fact the region

$$\left\{z \in U \mid |z| \geqslant 1, |\mathrm{Re}(z)| \leqslant \tfrac{1}{2}\right\}$$

is a fundamental region for the action of $G$, and looking at the way in which its sides are identified shows that $U/G$ is homeomorphic to a punctured sphere.

EXAMPLE 3. Consider the group of rationals $\mathbf{Q}$ acting on the real line $\mathbf{R}$ by addition. Taking the closure of $\mathbf{Q}$ in Homeo($\mathbf{R}$) gives a copy of $\mathbf{R}$ acting on itself by translation. Since $\mathbf{R}$ is path connected, $\mathbf{R}/\mathbf{Q}$ must be simply connected.

EXAMPLE 4. Let $G$ be a compact Lie group acting effectively on a simply connected space $X$. Assume that either $G$ is connected, or that $X^G$ (those points fixed by all elements of $G$) is nonempty. In both cases $G = \overline{G} = N$ and the orbit space $X/G$ must be simply connected.

EXAMPLE 5. Consider an irrational flow on the torus $T$. More precisely, let $\mathbf{R}$ act on $T \cong S^1 \times S^1$ as follows: the real number $r$ sends

$$\left(e^{2\pi ix}, e^{2\pi iy}\right) \quad \text{to} \quad \left(e^{2\pi i(x+r)}, e^{2\pi i(y+r\sqrt{2})}\right).$$

This action lifts to an action of $\pi_1(T) \times \mathbf{R}$ on $\mathbf{R}^2$ which has the same orbit space, namely $(m, n, r) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{R}$ sends $(x, y)$ to $(x + m + r, y + n + r\sqrt{2})$. One easily checks that the closure of this group of homeomorphisms of $\mathbf{R}^2$ is precisely the group of all translations of $\mathbf{R}^2$. Since this is a path connected group, the orbit space $T/\mathbf{R}$ is simply connected.

EXAMPLE 6. Let $F$ be a finitely generated free group and $X$ the graph of $F$ relative to a minimal set $S$ of generators. (That is to say $X$ has a vertex for each element of $F$, and an edge joins vertices $g$ and $h$ if and only if $h^{-1}g$ is an element of $S$ or the inverse of an element of $S$.) Note that $X$ is simply connected because $F$ is free. The action of $F$ on itself by left multiplication induces a free simplicial action of $F$ on $X$. If $G$ is any subgroup of $F$ we have an induced action of $G$ on $X$ and the orbit space $X/G$ is a one-dimensional simplicial complex. The fundamental group of this orbit space must therefore be free. But in terms of our result $G = \overline{G}$ and $N$ is the trivial subgroup, so this fundamental group is just $G$ and we recapture Nielsen's theorem that any subgroup of $F$ is free.

The conditions we shall need are listed below.

A. The projection $X \to X/\overline{G}$ has the path lifting property up to homotopy.

B. Given points $x, x' \in X$ plus a neighborhood $V$ of $x$ in $X$, if $x' \in \overline{G}x$ then $x' \in GV$.

C. The group $\overline{G}/N$ acts discontinuously on $X/N$.

(The technical terms mentioned above are defined as follows. A map $f\colon X \to Y$ has the path lifting property up to homotopy if given a path $\alpha\colon I \to Y$, and a point $p \in f^{-1}\alpha(0)$, we can find a path $\beta\colon I \to X$ such that $\beta(0) = p$, $f\beta(1) = \alpha(1)$ and

$f\beta \simeq \alpha$ rel$\{0, 1\}$. A group $G$ acts discontinuously on a space $X$ if each point $x$ of $X$ has a neighborhood $V$ such that $gV \cap V$ is empty unless $g$ fixes $x$.)

PROPOSITION 1. *If condition* B *is satisfied then* $X/G$ *and* $X/\overline{G}$ *have the same homotopy type.*

PROOF. Let $\pi$: $X \to X/G$ and $\phi$: $X/G \to X/\overline{G}$ denote the natural projections. Define $\psi$: $X/\overline{G} \to X/G$ as follows; given $z \in X/\overline{G}$ choose a point $y \in X/G$ which satisfies $\phi(y) = z$ and set $\psi(z) = y$. Then $\phi\psi$ is the identity map of $X/\overline{G}$.

We need to show $\psi\phi$ homotopic to the identity map of $X/G$. To this end define $F$: $X/G \times I \to X/G$ by

$$F(y, t) = \begin{cases} \psi\phi(y), & 0 \leqslant t < 1, \\ y, & t = 1. \end{cases}$$

If $F$ is continuous we are finished. Now given an open set $U$ of $X/G$ we have

$$F^{-1}(U) = \left[\phi^{-1}\psi^{-1}(U) \times [0, 1)\right] \cup \left[U \times \{1\}\right].$$

Clearly this is open in $X/G \times I$ if $U \subseteq \phi^{-1}\psi^{-1}(U)$, in other words if $\phi(U) \subseteq \psi^{-1}(U)$.

Suppose $z \in \phi(U)$, say $\phi(y) = z$ where $y \in U$. We need to check that $\psi(z)$ lies in $U$. Let $\psi(z) = y'$, then $\phi(y') = \phi(y) = z$. Choose $x, x' \in X$ with images $y, y'$ respectively in $X/G$. The set $V = \pi^{-1}(U)$ is an open neighborhood of $x$ in $X$, and $x' \in \overline{G}x$ since $x$ and $x'$ both map to $z$ in $X/\overline{G}$. By hypothesis we must have $x' \in GV$. Therefore $\pi(x')$ lies in $\pi(GV)$, that is to say $\psi(z)$ lies in $U$ as required.

COROLLARY 2. *If* $G$ *is a group of isometries of* $X$, *then* $X/G$ *and* $X/\overline{G}$ *have the same homotopy type.*

PROOF. Simply check condition B for a group of isometries. Suppose we have $x, x' \in X$ such that $x' \in \overline{G}x$, and let $V$ be a neighborhood of $x$ in $X$. Let $\varepsilon$ denote the distance from $x$ to $X - V$, and choose $g \in G$ such that $g(x)$ is within $\varepsilon$ of $x'$. Then, since $g$ is an isometry, $x' \in gV$ as required.

Condition A is the only one which is hard to check, so we list several situations where it is satisfied.

PROPOSITION 3. *The projection* $X \to X/\overline{G}$ *has the path lifting property up to homotopy if any one of the following holds.*

(a) *$G$ acts simplicially on a triangulation of $X$.*

(b) *The action of $G$ on $X$ is discontinuous, and the stabiliser of any point is finite.*

(c) *$\overline{G}$ is a compact Lie group.*

(d) *$\overline{G}$ is a locally compact Lie group and acts properly on $X$.*

(e) *$X/\overline{G}$ is semilocally simply connected.*

PROOF. (a), (b), (c), (d) can be found in **[1]**, **[2]**, **[3]**, **[4]**, respectively, and in these cases one can actually lift the given path, rather than just a path which is homotopic to it.

*Case* (e). We use an argument due to Smale [6], though our hypotheses are less restrictive. Let $\alpha\colon I \to X/\overline{G}$ be the given path, let $y_0 = \alpha(0)$ and suppose $x_0 \in \pi^{-1}(y_0)$. For each point $y$ of $X/\overline{G}$ choose a neighborhood $Uy$ such that loops in $Uy$ are null homotopic in $X/\overline{G}$. We can do this since $X/\overline{G}$ is semilocally simply connected. Given $x \in \pi^{-1}(y)$ use the local path connectedness of $X$, and the continuity of $\pi$, to find a path connected neighborhood $Py(x)$ of $x$ such that $\pi(Py(x)) \subseteq Uy$. If $x' \in \pi^{-1}(y)$ also, set $Py(x') = gPy(x)$ where $g$ is an element of $\overline{G}$ which sends $x$ to $x'$. Let $Vy = \pi(Py(x))$ and note that $Vy$ is a neighborhood of $y$ in $X/\overline{G}$ since $\pi$ is an open map.

Divide up $I$ into $N$ subintervals of equal length, taking $N$ sufficiently large so that $1/N$ is a Lebesgue number for the open covering $\{\alpha^{-1}(Vy) | y \in X/\overline{G}\}$ of $I$. Set $I_1 = \{t \in I | 0 \leqslant t \leqslant 1/N\}$ and note that $\alpha(I_1)$ must be contained in $Vy$ for some point $y \in X/\overline{G}$. Therefore $y_0 \in Vy$ and $x_0 \in Py(x)$ for some $x \in \pi^{-1}(y)$. Now $Py(x)$ is path connected, so we can join $x_0$ to some point of $Py(x) \cap \pi^{-1}(\alpha(1/N))$ by a path $\gamma\colon I \to Py(x)$. Define $\beta\colon I_1 \to X$ by $\beta(t) = \gamma(Nt)$. Then $\beta(0) = x_0$, $\pi\beta(1/N) = \alpha(1/N)$, and $\pi\beta \simeq \alpha|I_1 \operatorname{rel}\{0, 1/N\}$, since $\pi\beta$ and $\alpha|I_1$ have the same end points and both lie in $Uy$. This defines our lift (up to homotopy) $\beta$ over $I_1$; the remaining subintervals are dealt with in a similar manner.

THEOREM 4. *If $X$ is simply connected and if conditions* A, B *and* C *are satisfied, then $\pi_1(X/G)$ is isomorphic to $\overline{G}/N$.*

PROOF. We shall show that $X/N$ is simply connected. Assume this is done and note that $\overline{G}/N$ acts freely on $X/N$, because $N$ contains all the elements of $\overline{G}$ which have fixed points, and that it acts discontinuously by hypothesis. Therefore the projection $X/N \to X/\overline{G}$ is a covering map and we deduce $\pi_1(X/\overline{G}) \cong \overline{G}/N$. The theorem now follows from Proposition 1.

Choose a base point $p \in X$ and let $q = \pi(p)$, where $\pi$ now stands for the projection from $X$ to $X/N$. Define $\phi\colon N \to \pi_1(X/N, q)$ as follows. Given an element $g \in N$, join $p$ to $g(p)$ in $X$ by a path $\gamma$ and set

$$\phi(g) = \langle \pi \circ \gamma \rangle.$$

Notice that the choice of $\gamma$, amongst all paths joining $p$ to $g(p)$, is irrelevant since $X$ is simply connected. This function $\phi$ is a homomorphism, for given $g_1, g_2 \in N$, and having chosen $\gamma_1$ joining $p$ to $g_1(p)$ and $\gamma_2$ joining $p$ to $g_2(p)$, use $\gamma_1 \cdot g_1(\gamma_2)$ to join $p$ to $g_1 g_2(p)$. Then

$$\phi(g_1 g_2) = \langle \pi \circ (\gamma_1 g_1(\gamma_2)) \rangle = \langle \pi \circ \gamma_1 \rangle \langle \pi \circ \gamma_2 \rangle = \phi(g_1)\phi(g_2).$$

The kernel of $\phi$ is all of $N$. For if $g$ lies in the path component of the identity of $\overline{G}$, join $e$ to $g$ by a path $\{g_t | 0 \leqslant t \leqslant 1\}$ in $\overline{G}$. Then $\{g_t(p)\}$ joins $p$ to $g(p)$ in $X$ and projects to a single point in $X/N$. Therefore $g$ lies in the kernel of $\phi$. Now suppose $g$ is an element of $\overline{G}$ which fixes some point of $X$, say $g(x) = x$. Join $p$ to $x$ by a path $\gamma$, and use $\gamma(g\gamma)^{-1}$ to join $p$ to $g(p)$. This path projects to a null homotopic loop in $X/N$, so again $g$ lies in the kernel of $\phi$.

It only remains to show that $\phi$ is onto. Given an element of $\pi_1(X/N, q)$ represent it by a loop $\alpha$ based at $q$. Now $X \to X/\overline{G}$ has the path lifting property up to homotopy and, since $X/N \to X/\bar{G}$ is a covering map, so has $X \to X/N$. Therefore we can find a path $\beta$ in $X$ which begins at $p$ and which satisfies $\pi\beta(1) = q$, $\pi\beta \simeq \alpha$ rel$\{0, 1\}$. Since $\beta(1) \in \pi^{-1}(q)$ there is an element $g \in N$ such that $g(p) = \beta(1)$, and by construction $\phi(g) = \langle \pi \circ \beta \rangle = \langle \alpha \rangle$. This completes the proof.

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