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# ARTIN PRESENTATIONS FROM AN ALGEBRAIC VIEWPOINT

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Artin presentations are certain group presentations intimately related to pure braids and manifolds of dimensions two, three, and four. This paper studies combinatorial group theoretic properties of Artin presentations as interesting objects in their own right with subtle and pertinent problems.

*Keywords*: Artin presentation; combinatorial group theory; pure braid.

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## 1. Introduction

An *Artin presentation* is, by definition, a finite presentation:

$$r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

satisfying the following equation in $F_n$ (the free group on $x_1, \ldots, x_n$):

$$(r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n) \approx x_1 x_2 \cdots x_n.$$

Interesting group presentations often have roots in geometry or topology and Artin presentations are no exception. They were first studied by Emil Artin in 1925 in relation to his theory of braids [1, pp. 416–441]. González-Acuña, in 1975, coined the name Artin presentation and showed that they characterize the fundamental groups of closed, orientable 3-manifolds [8]. Their connection to 4-manifolds and knot theory was later revealed by Winkelnkemper [13].

These connections are deep, and will be discussed below to some extent. The main thesis of this paper is that Artin presentations are interesting objects in their own right. Their properties should be, whenever possible, determined and proven in a purely algebraic manner. For example, the Casson invariant in Artin presentation theory can be so obtained [4]. This is not an idle exercise, since pure algebraization is necessary in order to prepare applications of Artin presentation theory to, say, quantum computation, which in the guise of anyons also uses braid theory in a fundamental manner (see [6, 12]).

Let $\mathcal{R}_n$ denote the set of Artin presentations on $n$ generators $x_1, \ldots, x_n$. It is always assumed that the individual words $r_i$ are freely reduced in an Artin presentation.

Associated to an Artin presentation $r \in \mathcal{R}_n$ are

$$\pi(r) = \text{the group presented by } r, \text{ and}$$
$$A(r) = \text{the exponent sum matrix of } r.$$

That is, $A(r)$ is the $n \times n$ integer matrix whose $ij$th entry is the exponent sum of $x_i$ in $r_j$.

**Theorem 1.** *If $r$ is an Artin presentation, then $A(r)$ is symmetric.*

**Remark 1.** Every symmetric integer matrix appears as $A(r)$ for some Artin presentation $r$ (see [13, p. 248]).

Of course one defines the exponent sum matrix for any finite presentation, but in general, it need not be square and certainly not symmetric. This property of Artin presentations was first observed by Winkelnkemper [13] and was originally proved using the symplectic property of closed surface homeomorphisms. The new proof below proceeds directly from the definition of an Artin presentation and is entirely combinatorial group theoretic. The main ingredients in this proof are a technical result ($j$-reduction, see Sec. 3) and a combinatorial characterization of Artin presentations on two generators (making no mention of braids or automorphisms of $F_n$).

**Theorem 2.** *Artin presentations $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$ in $\mathcal{R}_2$ are characterized by*

$$r_1 = x_1^a (x_1 x_2)^c, \quad \text{and}$$
$$r_2 = x_2^b (x_1 x_2)^c, \quad \text{for some } a, b, c \in \mathbb{Z}.$$

Artin presentations occupy an interesting crossroad of discrete combinatorial group theory, pure braid theory, and low-dimensional manifold theory. For example, two of the deepest theorems in topology (and at opposite ends) percolate down to Artin presentations: the Jordan curve theorem and Donaldson's theorem. The former puts strong constraints on words that can appear as relations in Artin presentations.

**Theorem 3.** *If $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is an Artin presentation, then $r_i = x_i^{k_i} w_i$ for some integer $k_i$ and freely reduced word $w_i$ in $F_n$ satisfying: $w_i$ does not begin with a nonzero power of $x_i$, adjacent generators in $w_i$ are distinct and all generators appear to the power of $\pm 1$.*

While these conditions are restrictive, they are not sufficient (see Sec. 7).

At the other end of the spectrum, [13, Theorem I] states that if $r$ is an Artin presentation and $A(r)$ is definite but not congruent to $\pm I$ over $\mathbb{Z}$ then $\pi(r)$ is nontrivial. The statement of this theorem is purely group theoretic, although the proof relies heavily on differential geometric methods. This result reveals a close connection between Artin presentations and deeper properties of quadratic forms, a connection that simply does not exist with presentations in general (see Sec. 8).

This paper is organized as follows. Section 2 contains basic facts about free groups. Section 3 proves a technical result on $j$-reduction. Section 4 proves Theorem 2 characterizing Artin presentations on two generators. Section 5 proves that the exponent sum matrix of an Artin presentation is symmetric. Section 6 reviews some of the connections between Artin presentations and topology. Section 7 proves Theorem 3 on necessary conditions for words to appear as relations in an Artin presentation. Section 8 closes with some comments and open problems.

## 2. Preliminaries

This section recalls some basic facts about free groups and fixes some notation.

The free group $F_n = \langle x_1, \ldots, x_n \rangle$ is defined combinatorially in [9, Secs. 1.2 and 1.4]. It is common practice to abuse notation and write $w$ to mean both a word in the generators $x_1, \ldots, x_n$ and the equivalence class it represents in $F_n = \langle x_1, \ldots, x_n \rangle$. The context should make clear which is actually meant.

The solution of the word problem in $F_n$ is well known: perform simple free reductions on a given word $w$ in any order and as long as possible. A simple free reduction is the removal of $x_i^{\pm 1} x_i^{\mp 1}$ in $w$. A word is freely reduced if no simple free reduction is possible. Two words $w$ and $v$ represent the same element in $F_n$ if and only if they have identical free reductions. A purely combinatorial proof (using no topology) of this result is in [9, pp. 34–35]. Magnus, Karrass and Solitar give a concrete process, denoted $\rho$, for producing the unique free reduction $\rho(w)$ of a word $w$ in $F_n$. From here on $\rho$ denotes this process.

Given $u, v \in F_n$, write $u = v$ in case the words are identically equal when written out as products of $x_i^{\pm 1}$, $1 \le i \le n$, without performing any free reductions. A simple free insertion on a word $w$ in $F_n$ is the inverse process of a simple free reduction. Two words $u, v$ in $F_n$ are freely equal, written $u \approx v$, provided one can be obtained from the other by free reductions and insertions. Thus, the following are equivalent: $u$ and $v$ determine the same element in $F_n$, $u \approx v$, and $\rho(u) = \rho(v)$.

The definition of an Artin presentation can be rephrased using the notation above. Let $r_i$, $1 \le i \le n$, be freely reduced words in $F_n$. Then the presentation

$$r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

is an *Artin presentation* if and only if

$$\rho((r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n)) = x_1 x_2 \cdots x_n. \tag{AC}$$

This condition is referred to as (AC), the Artin condition.

**Remark 2.** Given $n$ words $r_i \in F_n$, $1 \le i \le n$, one easily checks if the Artin condition (AC) is satisfied simply by freely reducing. With large words, a computer algebra system such as MAGMA (where free reduction is automatic) is useful.

Substitutions on words in $F_n$ are performed as follows. Let $w$ be a word in $F_n$. Write $w = w(x_\mu)$ to emphasize that $w$ is a word in the letters $x_\mu$, $1 \le \mu \le n$. Let $y_\mu$, $1 \le \mu \le n$, be any expression. Then, let $w(y_\mu)$ denote the result of substituting $y_\mu$ for $x_\mu$ in $w(x_\mu)$. It is implicit that $y_\mu^{-1}$ is substituted for $x_\mu^{-1}$. Notice that no free reduction takes place in this definition, although removing appearances of 1 in a nontrivial expression is allowed.

The following basic properties will be used below.

(P1) Let $u_i$, $1 \le i \le k$, be any word in $F_n$. Then, $\rho\left(\prod_{i=1}^k u_i\right) = \rho\left(\prod_{i=1}^k \rho(u_i)\right)$.
(P2) Let $u$ be a freely reduced word in $F_n$ and suppose $u^{-1}x_i u \approx x_i$ for some $1 \le i \le n$, then $u = x_i^k$ for some integer $k$.
(P3) Let $w$ be a freely reduced word in $F_n$ and suppose that $w^{-1}(x_1x_2\cdots x_n)w \approx x_1x_2\cdots x_n$, then $w = (x_1x_2\cdots x_n)^k$ for some integer $k$.

Proofs of these properties follow for completeness. To prove (P1), note that $u_i \approx \rho(u_i)$ for $1 \le i \le k$ and so $\prod_{i=1}^k u_i \approx \prod_{i=1}^k \rho(u_i)$. The result follows by applying $\rho$ to this last equation. To prove (P2), note that the hypothesis implies $u = x_i^k v$ for some integer $k$ and some freely reduced word $v$ in $F_n$ where $v$ does not begin with a nonzero power of $x_i$, since this is the only way free reductions can take place in $u^{-1}x_i u$. Therefore $x_i \approx u^{-1}x_i u = v^{-1}x_i^{-k}x_i x_i^k v \approx v^{-1}x_i v$ where $v^{-1}x_i v$ is freely reduced, and so $x_i = v^{-1}x_i v$. This implies $v = 1$ and the result follows. Finally, to prove (P3), note that commuting elements in $F_n$ are powers of the same word [9, p. 42], and so $w \approx u^i$ and $x_1x_2\cdots x_n \approx u^j$ for some word $u$ in $F_n$ and integers $i$ and $j$. Without loss, assume $j > 0$ and $u$ is freely reduced. Write $u = v^{-1}Uv$ for the longest possible initial segment $v^{-1}$ of $u$ ($U$ must be nontrivial, but $v$ may be trivial). Then $x_1x_2\cdots x_n \approx u^j = (v^{-1}Uv)^j \approx v^{-1}U^j v$ where this last word is freely reduced ($u = v^{-1}Uv$ is freely reduced and $v^{-1}$ is the longest possible initial segment of $u$). Hence, $v = 1$, $j = 1$ and $u = x_1x_2\cdots x_n$. The result follows.

Property (P3) implies that if $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is an Artin presentation with all $r_i$ equal (i.e. $r_i = w$ for all $i$), then $w = (x_1x_2\cdots x_n)^k$ for some integer $k$.

Let $w$ be a word in $F_n$. The length of $w$, denoted $L(w)$, is the sum of the absolute values of the exponents of the generators appearing in $w$. The length of the trivial word $L(1)$ is zero. If $r$ is an Artin presentation, then $L(r)$ denotes the sum of the lengths of the words $r_i$ defining $r$.

## 3. $j$-Reduction

Given an Artin presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ in $\mathcal{R}_n$, $j$-reduction yields an Artin presentation on $n-1$ generators. The idea is that by deleting $r_j$, setting

$x_j = 1$ in the other $r_i$, freely reducing the individual resulting words and renumbering, one obtains an Artin presentation in $\mathcal{R}_{n-1}$. It was noted in [13, p. 251], that the result is in fact an Artin presentation by topological considerations. Below is a purely group theoretic proof of this fact. The renumbering step is omitted simply for notational reasons.

Fix $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ as an Artin presentation and $j$, an integer $1 \leq j \leq n$. Define

$$\begin{aligned} y_\mu &= x_\mu \quad \text{for } 1 \leq \mu \leq n \text{ and } \mu \neq j, \\ y_j &= 1, \\ u_i &= r_i(y_\mu) \quad \text{for } 1 \leq i \leq n \text{ and } i \neq j, \\ u_j &= 1, \quad \text{and} \\ s_i &= \rho(u_i) \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

**Lemma 1.** *With* $r$, $y_\mu$, $u_i$, *and* $s_i$ *as directly above,*

$$(s_1^{-1} x_1 s_1) \cdots (s_{j-1}^{-1} x_{j-1} s_{j-1})(s_{j+1}^{-1} x_{j+1} s_{j+1}) \cdots (s_n^{-1} x_n s_n) \approx x_1 \cdots x_{j-1} x_{j+1} \cdots x_n.$$

Notice that the free reductions required in the above equation occur in $F_{n-1} = \langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \rangle$ since no $x_j$ appear anywhere.

**Proof.** First, notice that if $w = w(x_\mu)$ is any word in $F_n$ then

$$\rho(w(y_\mu)) = \rho([\rho(w(x_\mu))](y_\mu)). \tag{$\star$}$$

Intuitively, this means that setting $x_j = 1$ in $w$ and then freely reducing produces exactly the same freely reduced word as freely reducing $w$, then setting all $x_j = 1$ and freely reducing again. To see this, let $F_{n-1} = \langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \rangle$ and define the homomorphism $\psi\colon F_n \to F_{n-1}$ by $x_i \mapsto x_i$, $i \neq j$, and $x_j \mapsto 1$. Since $w(x_\mu) \approx \rho(w(x_\mu))$, the well definition of $\psi$ implies that $\psi(w(x_\mu)) \approx \psi(\rho(w(x_\mu)))$. It follows that

$$\begin{aligned} w(y_\mu) &= \psi(w(x_\mu)) \\ &\approx \psi(\rho(w(x_\mu))) \\ &= [\rho(w(x_\mu))](y_\mu). \end{aligned}$$

Applying $\rho$ proves Eq. ($\star$).

Now, let $w(x_\mu) = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n)$. The Artin condition (AC) implies $\rho(w(x_\mu)) = x_1 x_2 \cdots x_n$. Thus

$$\begin{aligned} x_1 \cdots x_{j-1} x_{j+1} \cdots x_n &= \rho([x_1 x_2 \cdots x_n]\,(y_\mu)) \\ &= \rho([\rho(w(x_\mu))]\,(y_\mu)) \\ &= \rho(w(y_\mu)), \end{aligned}$$

where the last equality follows from Eq. ($\star$). Furthermore

$$\begin{aligned} w(y_\mu) &= \prod_{i=1}^{n}([r_i^{-1}x_ir_i](y_\mu)) \\ &= \left[\prod_{i=1}^{j-1}u_i^{-1}x_iu_i\right][u_j^{-1}1u_j]\left[\prod_{i=j+1}^{n}u_i^{-1}x_iu_i\right] \\ &\approx \prod_{i=1,\ i\neq j}^{n}u_i^{-1}x_iu_i. \end{aligned}$$

Applying $\rho$ and (P1) gives

$$\begin{aligned} \rho(w(y_\mu)) &= \rho\left(\prod_{i=1,\ i\neq j}^{n}\rho(u_i^{-1})x_i\rho(u_i)\right) \\ &= \rho\left(\prod_{i=1,\ i\neq j}^{n}s_i^{-1}x_is_i\right), \end{aligned}$$

and the result follows. □

## 4. Characterization of $\mathcal{R}_2$

This section shows that Artin presentations $r = \langle x_1, x_2 \mid r_1, r_2\rangle$ in $\mathcal{R}_2$ are characterized by

$$\begin{aligned} r_1 &= x_1^a(x_1x_2)^c, \quad \text{and} \\ r_2 &= x_2^b(x_1x_2)^c, \quad \text{for some } a, b, c \in \mathbb{Z}. \end{aligned}$$

An easy computation shows that these presentations satisfy the Artin condition (AC). To prove the converse, let $r = \langle x_1, x_2 \mid r_1, r_2\rangle$ be an Artin presentation in $\mathcal{R}_2$. The proof is by informal induction on $L(r) = L(r_1) + L(r_2)$. By definition, $r_1$ and $r_2$ are freely reduced. The cases $L(r) \leq 1$ are trivial, as are the cases where either $r_i = 1$ ($r_1 = 1 \Rightarrow x_1r_2^{-1}x_2r_2 \approx x_1x_2 \Rightarrow r_2^{-1}x_2r_2 \approx x_2 \Rightarrow r_2 = x_2^m$ by (P2)). So, assume each $r_i \neq 1$, in particular $L(r) \geq 2$.

Without loss of generality, each $r_i$ does not begin with a nonzero power of $x_i$. Otherwise, removing such a letter gives a shorter Artin presentation of the desired form by induction.

Write $r_1 = x_2^m w_1$ and $r_2 = x_1^n w_2$ for $|m|$ and $|n|$ positive integers as large as possible. Note the $w_i$ are freely reduced. Let $A = r_1^{-1}x_1r_1$ and $B = r_2^{-1}x_2r_2$, which are freely reduced. The Artin condition (AC) implies $AB \approx x_1x_2$. A simple induction shows that $L(A) = L(B)$ or $L(A) = L(B) \pm 2$. Hence, $L(r_1) = L(r_2)$ or $L(r_1) = L(r_2) \pm 1$. If $L(r_1) = L(r_2)$, then $AB \approx x_1x_2$ implies $r_1r_2^{-1} \approx 1$, which implies $r_1 \approx r_2$, a contradiction since the $r_i$ are freely reduced and begin with different letters. Without loss of generality, $L(r_1) = L(r_2) + 1$ (if $L(r_1) = L(r_2) - 1$ then take the inverse of both sides of (AC) and reindex, reducing to the "+" case).

Note that $m, n = -1$. As $AB \approx x_1x_2$, the last $L(r_2)$ letters in $A$ must cancel with $B$. This implies $r_1 = x_2^{\pm 1}r_2$ (so $|m| = 1$) and $x_1x_2 \approx (r_1^{-1}x_1x_2^{\pm 1})x_2r_2$. The sign must be "−" and $x_1x_2 \approx (r_1^{-1}x_1)r_2 = (r_2^{-1}x_2x_1)r_2 = (w_2^{-1}x_1^{-n}x_2x_1)x_1^nw_2$. This implies $n = -1$ as well.

Hence, $x_1x_2 \approx w_2^{-1}x_1x_2w_2$ and (P3) implies $w_2 = (x_1x_2)^k$ for some integer $k$. The case $k = 0$ satisfies (AC). Otherwise, $k < 0$ as $|n|$ was chosen as large as possible. In these cases, $r_1 = x_2^{-1}x_1^{-1}(x_1x_2)^k$ and $r_2 = x_1^{-1}(x_1x_2)^k$ satisfying (AC) as desired. This completes the proof of Theorem 2.

**Remark 3.** As pointed out by the referee, Theorem 2 has the following topological application. An Artin presentation $r \in \mathcal{R}_n$ determines a unique closed, orientable 3-manifold $M^3(r)$ as described in Sec. 6 ahead. An Artin presentation $r$ is also a presentation corresponding to a spine of $M^3(r)$; one may see this directly from the open book construction using the 2-cells $s_i \times [0, 1]$ and a little reflection, or one may use the special Heegaard decomposition of $M^3(r)$ determined by $r$ as described in [13, pp. 248–249]. Theorem 2 and [11, Theorems 2.1 and 3.1 (with a simple transformed presentation in 3.1)], immediately imply that if $r \in \mathcal{R}_2$, then $M^3(r)$ is either the connected sum of two lens spaces or is a Seifert fiber space over $S^2$ with at most three exceptional fibers. We remind the reader that in general, a 3-manifold is not uniquely determined by a spine. However, a closed, orientable 3-manifold is uniquely determined by an Artin presentation of its fundamental group. Moreover, the same data, namely an Artin presentation, also uniquely determines a smooth null cobordism of the 3-manifold, thus naturally tapping into 4D gauge theory.

## 5. Symmetry of $A(r)$

Let $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n\rangle$ be an Artin presentation. The symmetry of $A(r)$ will follow by induction on $n$. If $n = 1$ there is nothing to show and if $n = 2$ the result holds by the characterization in Theorem 2. So, assume $n \geq 3$. The idea is that $j$-reduction shows $A(r)$ is symmetric of the $j$th row and the $j$th column. Applying this three times with different values of $j$ gives the result. This is where the symmetry for $n = 2$ was needed as a base case.

Fix $j = n$ and define $y_\mu$, $u_i$, and $s_i$ as before Lemma 1 on $j$-reduction. Lemma 1 implies that $s = \langle x_1, \ldots, x_{n-1} \mid s_1, \ldots, s_{n-1}\rangle$ is an Artin presentation in $\mathcal{R}_{n-1}$ and so $A(s)$ is symmetric by induction. The key observation is that $[A(r)]_{i,k} = [A(s)]_{i,k}$ for $1 \leq i, k \leq n-1$. To see this, recall that $[A(r)]_{i,k}$ equals the exponent sum of $x_i$ in $r_k$. This sum also equals the exponent sum of $x_i$ in $u_k$ since $u_k$ is obtained from $r_k$ by setting $x_n = 1$. Further, $s_k = \rho(u_k)$ and each simple free reduction in passing from $u_k$ to $s_k$ preserves the exponent sum of each generator $x_\mu$. This shows that the upper left $(n-1) \times (n-1)$ block of $A(r)$ is symmetric. Repeating this process with $j = n-1$ and $j = n-2$ shows that $A(r)$ is symmetric, as desired.

## 6. Topology and Artin Presentations

The connections between Artin presentations and topology are well documented [3–5, 8, 13]. The interested reader is referred to these papers for detailed proofs of statements made below.

Artin presentations arise geometrically as follows. Let $\Omega_n$ denote the compact 2-disk with $n$ holes as in Fig. 1. The fundamental group $\pi_1(\Omega_n, p_0)$ is isomorphic to $F_n = \langle x_1, \ldots, x_n \rangle$ where $x_i$ is geometrically realized by a simple closed loop representing the class of $s_i \partial_i s_i^{-1}$. The generator $x_2$ is depicted in Fig. 2. Let $h$ be any self homeomorphism of $\Omega_n$ that is the identity on the boundary. Let $r_i = \rho(s_i h(s_i^{-1}))$. Then $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is an Artin presentation (see [3] for a detailed proof).

The converse is more interesting and was implicitly known to Artin [1]. Namely, if $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is an Artin presentation, then there corresponds a unique (up to isotopy rel $\partial\Omega_n$) self homeomorphism $h(r)$ of $\Omega_n$ that is the identity on $\partial\Omega_n$ (see [2, pp. 30–34; 3]). The group of such homeomorphisms is isomorphic to $P_n \times \mathbb{Z}^n$, where $P_n$ denotes the $n$ strand pure braid group. The $\mathbb{Z}^n$ central extension results from twisting the individual boundary components $\partial_i$, $1 \leq i \leq n$, by whole integer amounts. In this way, one sees that the set $\mathcal{R}_n$ of Artin presentations on $n$ generators is a group canonically isomorphic to $P_n \times \mathbb{Z}^n$. Note that the group composition law in $\mathcal{R}_n$ can be defined purely group theoretically with no mention of braids [13, p. 227].

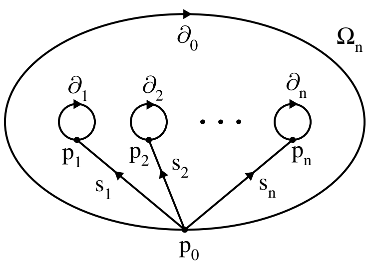


Fig. 1. $\Omega_n$ the compact 2-disk with $n$ holes, oriented boundary components $\partial_0, \ldots, \partial_n$ with basepoints $p_0, \ldots, p_n$, and oriented segments $s_i$ from $p_0$ to $p_i$.

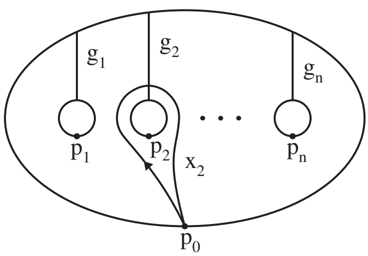


Fig. 2. Arcs $g_i$ and a generator $x_2$ of the fundamental group of $\Omega_n$.

Let $h$ be any homeomorphism of $\Omega_n$ that is the identity on the boundary. This map is completely determined up to isotopy (rel $\partial$) by the images $h(s_i)$ of the segments $s_i$ depicted in Fig. 1. Moreover, $h$ admits a unique smoothing up to isotopy (rel $\partial$) (see [3]). So, assume $h$ is smooth and the curves $h(s_i^{-1})$ intersect the segments $g_j$ transversely (see Fig. 2). A key observation is that the word $r_i = \rho(s_i h(s_i^{-1}))$ can be read off as follows: move along the oriented segment $h(s_i^{-1})$ beginning at $p_i$ and record $x_j$ ($x_j^{-1}$) each time the segment $g_j$ is crossed from left to right (respectively right to left). The reader should note that this gives a good reason to assume that the words $r_i$ in an Artin presentation are freely reduced. Namely, any appearance of $x_j^{\pm 1} x_j^{\mp 1}$ corresponds to the embedded segment $h(s_i^{-1})$ crossing $g_j$ and then, without crossing any other $g_k$, coming back and crossing $g_j$ in the opposite direction. These two crossings of $h(s_i^{-1})$ and $g_j$ bound segments in both $h(s_i^{-1})$ and $g_j$, and together they form a null homotopic simple closed curve. The 2D Schoenflies theorem implies that there is an isotopy of $h$ (rel $\partial$) that removes both of these crossings while fixing the rest of $h(s_i^{-1})$ and all of the other segments $h(s_k^{-1})$. Call a homeomorphism as above *tight* provided all such unnecessary crossings have been removed.

An Artin presentation $r$ determines a unique closed, orientable 3-manifold $M^3(r)$ by Winkelnkemper's open book construction [13, p. 246], as follows. Beginning with $r$, one obtains the homeomorphism $h(r)$ of $\Omega_n$ described above. Let $\Omega(h)$ denote the mapping torus of $h$, which is the quotient space $\Omega_n \times [0,1]/\sim$ where $(x, 0) \sim (h(x), 1)$ for all $x \in \Omega_n$. The boundary of $\Omega(h)$ is naturally identified with $\partial\Omega_n \times S^1$ (recall that $h$ is the identity on $\partial\Omega_n$) consisting of $n+1$ disjoint tori. Glue $\partial\Omega_n \times D^2$ onto $\Omega(h)$ by the identity on $\partial\Omega_n \times S^1$ and the result is $M^3(r)$. Note that $\Omega_n$ is called the *page* and $\partial\Omega_n$ is called the *binding*. Thus, $r$ also determines a canonical $(n+1)$-component link in $M^3(r)$ given by the binding. An unpublished result of González-Acuña (see [3] or [4] for a proof) states that: if $L$ is a link in a closed, orientable 3-manifold $M^3$, then $(M^3, L)$ is homeomorphic to $(M^3(r), K)$ for some Artin presentation $r$, where $K$ is the sublink $k_1, \ldots, k_m$ of the binding. An Artin presentation $r$ also determines a unique smooth, compact, simply-connected 4-manifold $W^4(r)$ with boundary $M^3(r)$ by a sort of relative open book construction [13, p. 250]. There are many interesting and open questions concerning these manifolds. The interested reader should see the references mentioned at the beginning of this section. We close this section with a few examples of Artin presentations that are interesting topologically.

**Example 1.** Let $r \in \mathcal{R}_3$ be given by

$$\begin{aligned} r_1 &= (x_3^{-1}, x_1 x_2^{-1} x_1^{-1}) \\ r_2 &= x_2(x_1, x_3) \\ r_3 &= x_1 x_2^{-1} x_1^{-1} (x_3, x_1) r_2 \end{aligned}$$

where $(x, y) = x^{-1} y^{-1} x y$. Then, $M^3(r)$ is the Heisenberg 3-manifold used by Goldman in [7] to show that not every 3-manifold has a conformally flat structure, thus providing a counterexample to a conjecture of Kuiper. The fundamental group $\pi(r)$ is isomorphic to the Heisenberg group presented by

$$\left\langle a, b, c \mid (a^{-1}, b^{-1}) = c, (a, c) = 1, (b, c) = 1 \right\rangle.$$

**Example 2.** Let $r \in \mathcal{R}_8$ be given by

$$\begin{aligned}
r_1 &= x_1^2 x_3^{-1} x_2 x_3 \\
r_2 &= x_2 x_3 x_1 x_3^{-1} x_2 x_3 \\
r_3 &= x_3 x_2 x_3 x_5^{-1} x_4 x_5 \\
r_4 &= x_4 x_5 (x_3, x_2) x_3 x_5^{-1} x_4 x_5 \\
r_5 &= x_5 x_4 x_5 x_6 x_7 x_8 x_7^{-1} (x_6, x_7) \\
r_6 &= (x_6 x_7)^2 (x_4 x_5 x_6 x_7 x_8)^{-1} r_5 \\
r_7 &= x_7 x_6 x_7 \\
r_8 &= x_8 (x_4 x_5 x_6 x_7)^{-1} x_5 x_4 x_5 x_6 x_7 x_8.
\end{aligned}$$

Then, $M^3(r)$ is the Poincaré homology 3-sphere with fundamental group $\pi(r) = I(120)$, the binary icosahedral group, and $A(r) = E_8$ (see Sec. 8), the matrix used by Milnor to construct his exotic 7-sphere [10, p. 174].

**Example 3.** In [5], Artin presentations $r \in \mathcal{R}_{22}$ are constructed with $M^3(r) = S^3$ and $W^4(r) \cup_\partial D^4$ the Kummer (or $K3$-) surface. Recall that the $K3$-surface is a quartic hypersurface in $\mathbb{C}P^3$. The shortest known Artin presentation for the $K3$-surface has total relator length 4398 [5, p. 83]. These presentations yield many interesting examples in 3- and 4-manifold topology (see [5, Sec. 3]).

## 7. The Words $r_i$

This section proves Theorem 3 on necessary conditions the defining words in an Artin presentation must satisfy. Namely, let $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ be an Artin presentation. The goal is to show that $r_i = x_i^{k_i} w_i$ for some integer $k_i$ and freely reduced word $w_i$ in $F_n$ such that: $w_i$ does not begin with a nonzero power of $x_i$, adjacent generators in $w_i$ are distinct and all generators appear to the power of $\pm 1$. Recall that each $r_i$ is freely reduced by definition.

Let $h$ be a smooth homeomorphism of $\Omega_n$ (fixed pointwise on $\partial\Omega_n$) corresponding to $r$ so that the curve $h(s_i^{-1})$ intersects the segment $g_j$ transversely. By discussions in the previous section, the map $h$ is assumed to be tight. The word $r_i$ is given by the segment $h(s_i^{-1})$ crossing the segment $g_j$ (see Figs. 1 and 2). Note that the embedded and oriented segment $h(s_i^{-1})$ starts at $p_i$ and ends at $p_0$. It is easy to see that $r_i$ can begin with an arbitrary power of $x_i$. Write $r_i = x_i^{k_i} w_i$ for $|k_i|$ as large as possible. The word $r_i$ is freely reduced and so $w_i$ is as well. Suppose $x_j^{+1}$ appears in $w_i$. This implies that $h(s_i^{-1})$ crosses $g_j$ from the left to the right. Now, if $x_j^{+1}$ appears next in $w_i$, then $h(s_i^{-1})$ would have to look like one of the two possibilities in Fig. 3. The former does not occur since, by the Jordan curve

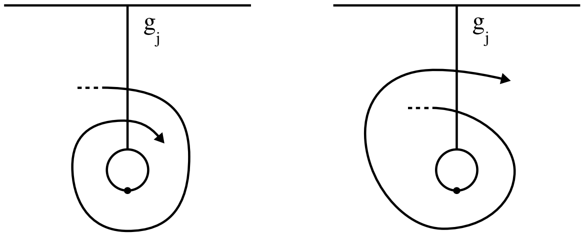


Fig. 3. Segment $h(s_i^{-1})$ wrapping around $\partial_j$ in two ways.

theorem, $h(s_i^{-1})$ cannot get out to end at $p_0$. The latter also gives a contradiction. By the Jordan curve theorem, $h(s_i^{-1})$ would have crossed $g_j$ from the right to the left (to get inside) giving $x_j^{-1}x_j$ in $r_i$. A similar argument applies to appearances of $x_j^{-1}$. This completes the proof of Theorem 3.

Notice that these necessary conditions are not sufficient. A simple counterexample is $\langle x_1, x_2 \mid 1, x_1 \rangle$ which plainly does not satisfy (AC). Nevertheless, the above restrictions on the defining relations in Artin presentations are strong.

**Remark 4.** In the spirit of this paper, one desires a combinatorial group theoretic proof of Theorem 3. Such a proof may be given using the relationship between Artin presentations and braid group automorphisms of $F_n$ [2, pp. 25, 30], along with a technical analysis of such automorphisms. Details will appear in a subsequent paper containing a study of the structure of words appearing as relations in Artin presentations.

## 8. Conclusion

As Magnus, Karrass and Solitar state [9, p. 8], "*Presentation theory attempts to derive information about a group from a presentation of it.*" This is a goal of the Artin presentation theory, where one further hopes to obtain information about the 3- and 4-manifolds determined by $r$.

The first basic observation is that $A(r)$ is a presentation matrix of the abelianization $\pi/[\pi,\pi]$ of $\pi(r)$. Namely, $A(r)$ is a linear map $\mathbb{Z}^n \to \mathbb{Z}^n$ and $\pi/[\pi,\pi]$ is isomorphic to $\mathbb{Z}^n/\operatorname{Im} A$. Thus, $\det A(r) \neq \pm 1$ is an abelian condition preventing $\pi(r)$ from being trivial. This condition applies equally well to all group presentations. However, Winkelnkemper's Theorem I [13, p. 240], described in the introduction above is a deeper abelian condition and is specific to Artin presentations. For example, let

$$\begin{aligned} s_1 &= x_1^2 x_2 & s_5 &= x_4 x_5^2 x_6 x_8 \\ s_2 &= x_1 x_2^2 x_3 & s_6 &= x_1 x_5 x_6 x_7 x_1^{-1} x_6 \\ s_3 &= x_2 x_3^2 x_4 & s_7 &= x_6 x_7^2 \\ s_4 &= x_3 x_4^2 x_5 & s_8 &= x_5 x_8^2. \end{aligned}$$

The (non-Artin) presentation $s = \langle x_1, \ldots, x_8 \mid s_1, \ldots, s_8 \rangle$ has exponent sum matrix equal to

$$E_8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}.$$

The matrix $E_8$ is well known: it is unimodular, even, positive definite, has signature 8, and is not congruent to $I$ over $\mathbb{Z}$. The group $\pi(s)$ presented by $s$ is the trivial group as MAGMA shows immediately (alternatively one may tinker with Tietze moves).

In stark contrast, let $r$ be any Artin presentation with $A(r) = E_8$ (e.g. $r$ from Example 2 in Sec. 6 above). Then, $\pi(r)$ cannot be trivial by Winkelnkemper's theorem [13, p. 240]. In fact, using the 3-manifold $M^3(r)$, one sees that the smallest $\pi(r)$ can be in this case is $I(120)$, the binary icosahedral group. Hence, with Artin presentations the deeper number theory of quadratic forms plays a real role in the groups so presented, unlike in the general case with arbitrary presentations of groups.

Does Winkelnkemper's theorem (or at least special cases of it) admit a purely group theoretic proof? What other abelian conditions exist preventing $\pi(r)$ from being trivial?

The word problem for groups admitting Artin presentations is another natural problem. Of course, González-Acuña's result that Artin presentations characterize the fundamental groups of closed, orientable 3-manifolds shows the relevance of this problem. The planarity of the page in the open book construction and its covering theory should be useful tools in studying this problem.

The restrictions placed on defining relations of Artin presentations by the Jordan curve theorem in the previous sections are substantial, yet not sufficient. What are natural (combinatorial group theoretic) sufficient conditions? The answer to this question may be relevant to the study of the faithfulness of the Gassner representation of the pure braid group (see [2, p. 133; 13, p. 266]).

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