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A characterization of sheaf-trivial, proper maps with cohomologically locally connected images

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Abstract

Let $f: X \to Y$ be a proper surjection of locally compact metric spaces. Throughout, the Leray sheafs of f are assumed to be (locally) trivial either in all dimensions or through a given dimension. Using a spectral sequence, the cohomological local connectivity of Y is analyzed and thus characterized by the structure of f. We define f to be *cohomologically locally connected* if, for each $y \in Y$, neighborhood U of y and $q \geqslant 0$, there is a neighborhood $V \subset U$ of y such that the image of the inclusion-induced homomorphism $H^q(f^{-1}(U)) \to H^q(f^{-1}(V))$ is finitely generated. The main result is:

Theorem. *If $f: X \to Y$ is a proper surjection of locally compact metric spaces and each Leray sheaf $\mathscr{H}^q[f]$ of f is locally constant, then any two of the following statements imply the third*:
(1) *Y is cohomologically locally connected.*
(2) *The stalk of $\mathscr{H}^q[f]$ is finitely generated, for all $q \geqslant 0$.*
(3) *f is cohomologically locally connected.*

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Let X and Y be locally compact metric spaces. This paper provides a characterization of proper surjections $f: X \to Y$ with locally constant Leray sheafs whose images are cohomologically locally connected. This characterization is similar to an earlier result of Dydak and Walsh [3], wherein they determine sufficient conditions for the local cohomological connectivity of X to be preserved by f. A principal application of their results is, in essence, that if f is a shape fibration with X cohomologically locally connected (clc), then Y is necessarily clc. Their characterization is well suited to that purpose. The characterization presented here is more

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simply stated and is not as restrictive. Consider the following example: let W be the Warsaw circle, X the product of W with the Euclidean line Y, and f the natural projection of X onto its Euclidean factor. Then X is not clc (so the result of Dydak and Walsh does not apply), but the image of f is. Now f has the property that for all $y \in Y$ there are neighborhoods $V \subset U$ of y such that the image of the inclusion-induced homomorphism $H^q(f^{-1}(U)) \to H^q(f^{-1}(V))$ (Čech cohomology) has finitely generated image for all $q \geqslant 0$; we will abbreviate this statement by saying that f is clc. In certain instances, this is enough to determine that f has clc image.

Theorem. *If $f: X \to Y$ is a proper surjection with locally constant Leray sheaves between locally compact metric spaces such that, for all $y \in Y$ and $q \geqslant 0$, $H^q(f^{-1}(y))$ is finitely generated, then f is clc if and only if Y is clc.*

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1. Definitions

All spaces are assumed to be locally compact metric spaces. All cohomology will be as in the Čech theory. Let $\Lambda = \{2, 3, \dots\} \cup \{\infty\}$; we use this often as an index set. We regard Λ as a totally ordered set, where ∞ represents the maximal element. Let R be any PID.

For the definition of a stack (presheaf), or sheaf, on a space Y and most of the following terminology, see [7] or [2]. Let $\mathscr{H}^q[f; R]$ be the Leray sheaf of f. When, by context, either f or R is understood, we may simple write $\mathscr{H}^q$. A sheaf is *trivial* if it is equivalent, in the category of sheaves, to a constant sheaf $M \times Y$, where M is an R-module; a sheaf $\mathscr{S}$ is *locally trivial* if for each $y \in Y$ there is a neighborhood U of y in Y such that $\mathscr{S}|_U$ is trivial.

There are three definitions of sheaf cohomology (categorical, Alexander and Čech) of a space, all of which agree for a given sheaf and the type of supports we consider; thus we shall speak unambiguously of *the* cohomology with coefficients in the Leray sheaf. If the space is, for example, an ANR and the sheaf is trivial, then each of these theories agrees with the singular theory [7].

We say that f is *sheaf-trivial over R in dimension k* [respectively *locally sheaf-trivial over R in dimension k*], denoted *k-ST_R* [respectively *k-LST_R*], if $\mathscr{H}^k[f; R]$ is trivial [respectively locally trivial] over Y. We say that f is sheaf-trivial over R *through* dimension k [respectively locally sheaf-trivial over R *through* dimension k], denoted ST_R^k [respectively LST_R^k], if f is *l-ST_R* [respectively *l-LST_R*] for all $0 \leqslant l \leqslant k$.

Given a surjective map $f: X \to Y$, for $A \subseteq Y$ we will follow the notation of James [4] and let $X_A = f^{-1}(A)$; if $y \in Y$ then $X_y = f^{-1}(y)$. Since the maps we discuss are closed, proper surjections, we remark here that it is well known that we may identify each stalk of $\mathscr{H}^*[f; R]$ over a point $y \in Y$ with $H^*(X_y; R)$; in this situation, the collection $\{X_C \mid C$ is a closed neighborhood of $y\}$ is a cofinal system of closed neighborhoods of X_y to which we may now apply the continuity of Čech cohomology.

A space Y has *finite cohomological type* (*over* R) [*respectively through dimension* k] if $H^q(Y; R)$ is finitely generated as an R-module for any q [respectively $q \leqslant k$]. A map $f: X \to Y$ has *finite cohomological type* (*over* R) [*through dimension* k] if X_y has finite cohomological type (over R) [through dimension k] for all $y \in Y$.

A space Y [respectively a map $f: X \to Y$] is *cohomologically locally connected in dimension i with respect to* R, denoted *i-clc*$_R$, if for all $y \in Y$ and all neighborhoods U of y in Y, there is a neighborhood $V \subseteq U$ of x such that $H^i(U; R) \to H^i(V; R)$ [respectively $H^i(X_U; R) \to H^i(X_V; R)$] has finitely generated image. The definition of a map being clc is, to this author's knowledge, novel; using James's terminology [4] we could instead describe X as being fibrewise cohomologically locally connected, but we choose to emphasize the role of f.

Proposition 1.1. *Let $f: X \to Y$ be a proper, clc_R^k map between locally compact metrizable spaces. Then f has finite cohomological type over R through dimension k.*

Proof. Let $y \in Y$. Since f is proper and X and Y are locally compact metric spaces, f is closed. Thus, $\{f^{-1}(V) \mid V$ is a neighborhood of y in $Y\}$ is a cofinal subset of $\{W \mid W$ is a neighborhood of $f^{-1}(y)$ in $X\}$. Continuity of Čech cohomology yields the concluding statement. □

The following theorem's proof is found in [2, pp. 140–141]. The reader is also referred to [1] as an excellent and enjoyable guide to understanding the Leray sheaf, sheaf cohomology and the structure of this spectral sequence.

Theorem 1.2 (Leray–Grothendieck). *For a closed map $f: X \to Y$, there is a first quadrant spectral sequence*

$$E_2^{p,q} = H_\phi^p(Y; \mathscr{H}^q[f; R]) \Rightarrow H_{\phi(\psi)}^{p+q}(X; R).$$

Remarks 1.3. (1) For our purposes we need only consider the family ψ of closed supports on X, and ϕ the family of closed supports on Y. In this case $\phi(\psi) = \psi$.

(2) $E_{r+1} = \ker(d_r)/\mathrm{im}(d_r)$, where the differential $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1}$ has bidegree $(r, 1-r)$.

(3) We define $Z_r^{p,q} = \ker\, d_r^{p,q}$ and $B_r^{p,q} = \mathrm{im}\, d_r^{p,q}$. Let $Z_\infty^{p,q} = \bigcap_{r \geqslant 2} Z_r^{p,q}$ and $B_\infty^{p,q} = \bigcup_{r \geqslant 2} B_r^{p,q}$. Then $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$.

(4) By the third isomorphism theorem, we may assume

$$0 \subset B_2^{p,q} \subset \cdots \subset B_r^{p,q} \subset B_{r+1}^{p,q} \subset \cdots \subset Z_{r+1}^{p,q} \subset Z_r^{p,q} \subset \cdots \subset Z_2^{p,q} \subset E_2^{p,q}.$$

(5) $E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots = E_\infty^{p,q}$ for $r \geqslant p + q + 1$.

(6) There is a filtration $0 \subset J_0 \subset J_1 \subset \cdots \subset J_p = H^p_{\phi(\psi)}(X)$, where $J_0 = E_\infty^{p,0}$ and $J_i/J_{i-1} = E_\infty^{p-i,i}$, $i \leqslant p$.

We also make use of the following lemma.

Lemma 1.4 [2, p. 77]. *Let the following diagram denote a commutative diagram of R-modules, where each of the vertical maps ρ, σ has finitely generated image, and the middle row is exact. Then the image of L_1 under the map $\tau_2 \circ \tau_1$ is finitely generated.*

$$\begin{array}{ccccc} & & L_1 & \longrightarrow & L_2 \\ & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \sigma} \\ M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tau_2} & & \\ N_1 & \longrightarrow & N_2 & & \end{array}$$

2. Principal results

In this section, we characterize local cohomological connectivity of the image of a sufficiently (locally) sheaf-trivial map. Standing hypotheses for this section are that $f: X \to Y$ is a proper surjection between locally compact metric spaces, R is a PID and the cohomology theory is Čech (with coefficients in R unless explicitly stated otherwise).

The following is the converse to the main Theorem. Its proof is given first as some of the notation and arguments of the main Theorem can be given with greater clarity here.

Theorem 2.1. *Suppose f is LST_R^k, f has finite cohomological type through dimension k and Y is clc_R^k. Then f is clc_R^k.*

Proof. Let y be any element of Y. Let $k \geqslant 0$. Let U be any neighborhood of y; since Y is 0-clc_R, we assume U to be connected. We may assume that $f \mid X_U$ is ST_R^k. We will show that for all l, where $k \geqslant l \geqslant 0$, there exists a connected neighborhood V of y such that $V \subseteq U$ and the image of the inclusion-induced homomorphism $H^l(X_U) \to H^l(X_V)$ is finitely generated as an R-module.

Under the current hypotheses, we may find a connected neighborhood $V \subseteq Q \subseteq U$ of y such that the images of $H^l(U) \to H^l(Q)$ and $H^l(Q) \to H^l(V)$ are finitely generated for $k \geqslant l \geqslant 0$; if desired, we may replace "finitely generated" by "trivial", since Y is locally compact, Čech cohomology is continuous and any generator of a finitely generated image has compact support.

There is a natural, inclusion-induced morphism from the map $f \mid_{X_B}$ to the map $f \mid_{X_U}$; similarly for $f \mid_{X_V}$ to the map $f \mid_{X_Q}$—in fact, we ask the reader to keep in mind that the following statements will hold for the pair (Q, V) as well as the pair (U, Q). The spectral sequence functor is contravariant and so induces a natural map from $E(U)$, the spectral sequence of $f \mid_{X_U}$, to $E(Q)$, the spectral sequence of $f \mid_{X_Q}$. Needing to distinguish terms and differentials of both sequences, let $E_r^{p,q}(W)$, $d_r^{p,q}(W)$, $Z_r^{p,q}(W)$, and $B_r^{p,q}(W)$ denote those entities arising from the particular proper map $f \mid_{X_W}$ (where $W = U$, Q, or V) which correspond to those described in Remarks 1.3.

For $0 \leqslant q \leqslant k$ and $W = U$ or Q, let $\mathscr{H}^q(W) = \mathscr{H}^q[f \mid_{X_W}]$. Since $f \mid_{X_U}$ is ST_R^k and $U \supset Q$ is connected, the module of sections of $\mathscr{H}^q$ over Q is isomorphic to $H^q(X_y) \cong H^0(Q; \mathscr{H}^q(Q)) = E_2^{0,q}(Q)$, which by hypothesis is finitely generated for $q \leqslant k$. We can now apply Lemma 1.4 to the commutative diagram, obtained by applying the naturality of the Universal Coefficient theorem:

$$\begin{array}{ccccc}
 & & H^p(U; \mathscr{H}^q(U)) & \longrightarrow & H^{p+1}(U) * \mathscr{H}_y^q \\
 & & \downarrow & & \downarrow \\
H^p(Q) \otimes \mathscr{H}_y^q & \longrightarrow & H^p(Q; \mathscr{H}^q(Q)) & \longrightarrow & H^{p+1}(Q) * \mathscr{H}_y^q \\
\downarrow & & \downarrow & & \\
H^p(V) \otimes \mathscr{H}_y^q & \longrightarrow & H^p(V; \mathscr{H}^q(V)) & &
\end{array}$$

We thus conclude that the image of $E_2^{p,q}(U) = H^p(U; \mathscr{H}^q(U)) \to H^p(V; \mathscr{H}^q(V)) = E_2^{p,q}(V)$ is finitely generated.

Now, fix p and q, where $0 \leqslant p, q \leqslant k$. We argue that $Z_r^{p,q}(U) \to Z_r^{p,q}(V)$ has finitely generated image. Recall that $Z_r^{p,q}(U)$ includes into $E_2^{p,q}(U)$ (see Remark 1.3(3)). So, in fact, the image in question is equal to the image of the composition $Z_r^{p,q}(U) \to E_2^{p,q}(U) \to E_2^{p,q}(V)$, where the latter map has finitely generated image; since R is a PID, our image must indeed be finitely generated. In particular, we may now conclude that $Z_\infty^{p,q}(U) \to Z_\infty^{p,q}(V)$, and hence $E_\infty^{p,q}(U) \to E_\infty^{p,q}(V)$, has finitely generated image for $0 \leqslant p, q \leqslant k$.

Using the technique outlined in the above two paragraphs, we find a sequence $U = U_0 \supset U_1 \supset \cdots \supset U_{k+1}$ of neighborhoods of y, so that, when $1 \leqslant i \leqslant k$, the image of $E_\infty^{p,q}(U_{i-1}) \to E_\infty^{p,q}(U_i)$, is finitely generated for $0 \leqslant p, q \leqslant k$. Recall that, for any fixed $l \geqslant 0$, there is a filtration $J_0(W) \subseteq J_1(W) \subseteq \cdots \subseteq J_l(W) = H^k(X_W)$ with $J_0(W) = E_\infty^{l,0}(W)$ and $J_i(W)/J_{i-1}(W) = E_\infty^{i,l-i}(W)$ for all $1 \leqslant i \leqslant l$, where $W = U_j$ and $0 \leqslant j \leqslant k+1$. Fix l between 0 and k, and select for each U_j its associated filtration. Then, for $1 \leqslant i \leqslant k$ the (inclusion-induced) images of $J_0(U_i) \to J_0(U_{k+1})$ and of $J_j(U_i)/J_{j-1}(U_i) \to J_j(U_{i+1})/J_{j-1}(U_{i+1})$ are finitely generated for $1 \leqslant j \leqslant l$.

We now claim that the inclusion-induced image $J_j(U_{k-j}) \to J_j(U_{k+1})$, is finitely generated for $0 \leqslant j \leqslant l$. We prove this by induction on j. The base case $j = 0$ is true by construction of U_k and U_{k+1}. Suppose the statement is true for $j-1$ where $0<j \leqslant l$. We now apply Lemma 1.4 to the commutative diagram:

$$\begin{array}{ccccc}
 & & J_j(U_{k-j}) & \longrightarrow & J_j(U_{k-j})/J_{j-1}(U_{k-j}) \\
 & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \sigma} \\
J_{j-1}(U_{k-(j-1)}) & \longrightarrow & J_j(U_{k-(j-1)}) & \longrightarrow & J_j(U_{k-(j-1)})/J_{j-1}(U_{k-(j-1)}) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tau_2} & & \\
J_{j-1}(U_{k+1}) & \longrightarrow & J_j(U_{k+1}) & &
\end{array}$$

Since σ has finitely generated image by construction of U_{k-j+1} and U_{k-j}, and ρ has finitely generated image by the inductive hypothesis, the image of the composition of the inclusion-induced maps τ_1 and τ_2 is finitely generated. Thus, the claim is proved.

In particular, we now have established that $H^l(X_{U_{k-l}}) \to H^l(X_{U_{k+1}})$ has finitely generated image. By virtue of the inclusion-induced composition $H^l(X_{U_0}) \to H^l(X_{U_{k-l}}) \to H^l(X_{U_{k+1}})$ we arrive at the conclusion that f is l-clc$_R$ for any $l \leqslant k$. $\square$

We now state and prove the critical result.

Theorem 2.2. *Suppose f is LST$_R^{k-1}$ and f is clc$_R^k$. Then Y is clc$_R^k$.*

Proof. It is sufficient to prove the case where f is ST$_R^{k-1}$. For if f is LST$_R^{k-1}$, given $y \in Y$, we may find for each l $(0 \leqslant l \leqslant k-1)$ a neighborhood U_l of y such that $\mathscr{H}^l[f \mid_{f^{-1}(U_l)}]$ is trivial; thus $f \mid_{X_U}$ is ST$_R^{k-1}$, where $U = \bigcap_{i=1}^{k-1} U_i$. Let us also note at this point that $\mathscr{H}^0$ is sheaf-isomorphic to $U \times R \times A$ where A is a set having the cardinality of the number of components of U_y; since f is clc^0, the set A is finite. Thus, by the Universal Coefficient theorem [2], $H^l(U; \mathscr{H}^0)$ is a finite direct sum of modules isomorphic to $H^l(U)$. In particular, $H^l(U)$ is isomorphic to a submodule of $H^l(U; \mathscr{H}^0)$.

Let $y \in Y$ and U be a neighborhood of y in Y. Since Y is clc$_R^k$, we may assume, without loss of generality, that U is connected. Consider the spectral sequence of the map $f \mid_{X_U}$. Let ϕ be the system of closed supports in U, ψ those in X_U. We use the notation of the previous proof when restricting the map, sheaves, supports and spectral sequences to subsets of U; if the argument of neighborhood is omitted, then we are referring to the entity related to U: for example, $E_r^{p,q} = E_r^{p,q}(U)$.

Claim. *There is a neighborhood $V \subset U$ of y such that the inclusion-induced image $\operatorname{im}\{E_r^{p,q}(U) \to E_r^{p,q}(V)\}$ is finitely generated for all $q<k$, $0 \leqslant p+q \leqslant k$ and $r \in \Lambda = \{2, 3, \ldots\} \cup \{\infty\}$.*

It is sufficient to prove this claim, for in particular it will show that $\operatorname{im}\{H^l(U; \mathscr{H}^0) = E_2^{l,0}(U) \to E_2^{l,0}(V) = H^l(V; \mathscr{H}^0)\}$ is finitely generated for $0 \leqslant l \leqslant$

k; by the comment at the end of the first paragraph in this proof, we may in turn conclude that Y is clc_R^k.

Towards establishing the claim, first note that $E_2^{0,q} = H_\Phi^0(U; \mathscr{H}^q)$ is naturally isomorphic to the module of sections of the trivial sheaf $\mathscr{H}^q$ over U which, by Proposition 1.1 and hypothesis, is finitely generated when $k > q \geqslant 0$. Moreover, $E_r^{0,q} = \ker d_{r-1}^{o,q} \subseteq E_{r-1}^{0,q} = \ker d_{r-2}^{o,q} \subseteq \cdots \subseteq E_2^{0,q}$ shows that $E_r^{0,q}$ is finitely generated when $k > q \geqslant 0$, $r \geqslant 2$; since $E_\infty^{0,q} \cong \ker d_{q+1}^{o,q} = E_{q+1}^{0,q}$ (the spectral sequence is in the first quadrant), it follows that $E_\infty^{0,q}$ is finitely generated as well.

We now prove the rest of the claim by strong induction on p. Let $\Sigma(p)$ denote the statement: "There is a neighborhood $V_p \subset U$ of y such that $\mathrm{im}\{E_r^{p',q}(U) \to E_r^{p',q}(V_p)\}$ is finitely generated, for all $p' \leqslant p$, $k > q \geqslant 0$, $p' + q \leqslant p$, and $r \in \Lambda$." The previous paragraph shows that $\Sigma(0)$ is true ($V_0 = U$). Assuming $\Sigma(p-1)$ to be true, we establish that $\Sigma(p)$ is true, where $0 < p \leqslant k$. Let V_{p-1} be the neighborhood guaranteed by $\Sigma(p-1)$. By hypothesis, there exists $V_p' \subset V_{p-1}$, a neighborhood of y such that $\mathrm{im}\{H^l(X_{V_{p-1}}) \to H^l(X_{V_p'})\}$ is finitely generated, for all $0 \leqslant l \leqslant k$.

In particular, we first argue that $\mathrm{im}\{E_r^{p,0}(U) \to E_r^{p,0}(V_p')\}$ is finitely generated for all $r \in \Lambda$. Note that, by hypothesis, the composition $H^p(X_U) \to H^p(X_{V_{p-1}}) \to H^p(X_{V_p'})$ has finitely generated image. Since $E_\infty^{p,0}$ is a submodule of $H^p(X_U)$ given by the filtration induced by the spectral sequence, we may look at the exact commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E_\infty^{p,0}(U) & \longrightarrow & H^p(X_U) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_\infty^{p,0}(V_p') & \longrightarrow & H^p(X_{V_p'}) \end{array}$$

and conclude that the image of $E_\infty^{p,0}(U) \to E_\infty^{p,0}(V_p')$ is finitely generated. Also note that for any $W \subset Y$ we have $d_{p+1}^{-1,p}(W) = 0$, implying $\ker d_{p+1}^{p,0} = E_{p+1}^{p,0} \cong E_{p+2}^{p,0} \cong E_{p+3}^{p,0} \cong \cdots \cong E_\infty^{p,0}$. Thus the image of $E_r^{p,0}(U) \to E_r^{p,0}(V_p')$ is finitely generated for $r \geqslant p+1$; to show that we may find a neighborhood of y such that this is true for $2 \leqslant r \leqslant p$ as well, we perform another induction.

To this end, let $\Phi(i)$ denote the statement "There is a neighborhood $W_i \subset V_p'$ of y such that the image of $E_{p+1-i}^{p,0}(U) \to E_{p+1-i}^{p,0}(W_i)$ is finitely generated"; note that $i \leqslant p-1 < k$ is a must. Also, $\Phi(0)$ is true with $W_0 = V_p'$, as shown in the previous paragraph. Assume $\Phi(i)$ to be true and $i < p-1$. We combine three pertinent observations:

(1) $Z_{p-i}^{p,0} = E_{p-i(=p+1-(i+1))}^{p,0}$;

(2) by $\Sigma(p-1)$ and the fact that $i < p \leqslant k$ and $p-i-1 < p$, we may choose a neighborhood $W_{i+1} \subset W_i$ of y such that the image of $B_{p-i}^{i,p-i-1}(W_i) \to B_{p-i}^{i,p-i-1}(W_{i+1})$ is finitely generated;

(3) $E_{p+1-i}^{p,0}(U) = Z_{p-i}^{p,0}(U)/B_{p-i}^{i,p-i-1}(U) \to E_{p+1-i}^{p,0}(V_p') = Z_{p-i}^{p,0}(V_p')/B_{p-i}^{i,p-i-1}(V_p')$ has finitely generated image, by $\Phi(i)$.

We then apply Lemma 1.4 to the following commutative diagram:

$$\begin{array}{ccccc}
 & & Z^{p,0}_{p-i}(U) & \longrightarrow & E^{p,0}_{p+1-i}(U) \\
 & & \downarrow & & \downarrow \\
B^{i,p-i-1}_{p-i}(W_i) & \longrightarrow & Z^{p,0}_{p-i}(W_i) & \longrightarrow & E^{p,0}_{p+1-i}(W_i) \\
\downarrow & & \downarrow & & \\
B^{i,p-i-1}_{p-i}(W_{i+1}) & \longrightarrow & Z^{p,0}_{p-i}(W_{i+1}) & &
\end{array}$$

So the image of $E^{p,0}_{p+1-(i+1)}(U) \to E^{p,0}_{p+1-(i+1)}(W_{i+1})$ is finitely generated, i.e., we have shown that $\Phi(i+1)$ is true. Note that $V_p' = W_0 \supset W_1 \supset \cdots \supset W_{p-1}$.

This induction now yields that, for our fixed p, the image of $E^{p,0}_r(U) \to E^{p,0}_r(W_{p-1})$ is finitely generated for all $r \in \Lambda$. For our fixed p, we go through the above arguments and also apply $\Sigma(p-1)$ to construct a neighborhood $V_p \subset W_{p-1}$ of y such that $E^{p',0}_r(W_{p-1}) \to E^{p',0}_r(V_p)$ has finitely generated image for all $r \in \Lambda$ and $p' \leqslant p$. Note that when $r = 2$, we may conclude that the image of $H^{p'}(W_{p-1}) \to H^{p'}(V_p)$ is finitely generated for $p' \leqslant p$.

Now for a final induction to establish $\Sigma(p)$. For $2 \leqslant r$, let $\Omega(r)$ denote the statement: "The image of $E^{p',q}_r(U) \to E^{p',q}_r(V_p)$ is finitely generated, for $k > q \geqslant 0$ and $p' + q \leqslant p$."

Now, $E^{p,0}_2(U) = H^p(U; \mathscr{H}^0) \to E^{p,0}_2(W_{p-1}) = H^p(W_{p-1}; \mathscr{H}^0)$ has finitely generated image, by $\Phi(p-1)$. Also consider $E^{p',q}_2 = H^{p'}(U; \mathscr{H}^q)$, where $0 < q < k$ and $p' + q \leqslant p$. The Universal Coefficient theorem (for Čech cohomology), which applies by the sheaf-triviality through dimension $k-1$ of f, states that

$$0 \to \left(H^{p'}(U) \otimes \mathscr{H}^q_y\right) \to H^{p'}(U; \mathscr{H}^q) = E^{p',q}_2 \to \left(H^{p'+1}(U) * \mathscr{H}^q_y\right) \to 0$$

is exact [6, p. 338]. Along with the first statement of this paragraph, the hypothesis that $\mathscr{H}^q_y$ is finitely generated and the naturality of the short exact sequence, we note that $\Sigma(p')$ implies that we can apply Lemma 1.4 to the following exact commutative diagram:

$$\begin{array}{ccccc}
 & & H^{p'}(U; \mathscr{H}^q) & \longrightarrow & H^{p'+1}(U) * \mathscr{H}^q_y \\
 & & \downarrow & & \downarrow \\
H^{p'}(W_{p-1}) \otimes \mathscr{H}^q_y & \longrightarrow & H^{p'}(W_{p-1}; \mathscr{H}^q) & \longrightarrow & H^{p'+1}(W_{p-1}) * \mathscr{H}^q_y \\
\downarrow & & \downarrow & & \\
H^{p'}(V_p) \otimes \mathscr{H}^q_y & \longrightarrow & H^{p'}(V_p; \mathscr{H}^q) & &
\end{array}$$

Hence, $E^{p',q}_2(U) \to E^{p',q}_2(V_p)$ is finitely generated. So $\Omega(2)$ is true.

Suppose $\Omega(r-1)$ is true. First note, for p' and q such that $0 \leqslant q < k$ and $p' + q \leqslant p$, that $(p' - r + 1) + (q + r - 2) < p \leqslant k$ and so $q + r - 2 < k$. So the image of $Z^{p',q}_{r-1}(U) \to Z^{p',q}_{r-1}(V_p)$ is finitely generated, by $\Omega(r-1)$; similarly, the image of $B^{p'-r+1,q+r-2}_{r-1}(U) \to B^{p'-r+1,q+r-2}_{r-1}(V_p)$ is finitely generated. It should be made clear to the reader at this point that the quantity $p'-r+1$ will indeed be negative for $r>p'+1$, but in this case the differential $d_{r-1}^{p'-r+1,q+r-2}$ and its image are trivial. We now observe from the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_{r-1}^{p'-r+1,q+r-2}(U) & \longrightarrow & Z_{r-1}^{p',q}(U) & \longrightarrow & E_r^{p',q}(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_{r-1}^{p'-r+1,q+r-2}(V_p) & \longrightarrow & Z_{r-1}^{p',q}(V_p) & \longrightarrow & E_r^{p',q}(V_p) & \longrightarrow & 0 \end{array}$$

that the image of $E_r^{p',q}(U)\to E_r^{p',q}(V_p)$ is finitely generated. So $\Omega(r)$ is true. By induction, $E_r^{p',q}(U)\to E_r^{p',q}(V_p)$ is finitely generated when $k>q\geqslant 0$, $r\geqslant 2$, and $p'+q\leqslant p$.

At last, given $k>q\geqslant 0$ and $p'\leqslant p-q$, then $E_\infty^{p',q}\cong E_{p'+q+2}^{p',q}$; so in this case $E_\infty^{p',q}(U)\to E_\infty^{p',q}(V_p)$ has finitely generated image. Thus $\Sigma(p)$ is true.

By induction, $\Sigma(p)$ is true for all $k\geqslant p\geqslant 0$. Also note that, by construction, we can have $U\supset V_1\supset\cdots\supset V_k$. In particular, the image of $H^p(U;\mathscr{H}^0)=E_2^{p,0}(U)\to E_2^{p,0}(V_k)=H^p(V_k;\mathscr{H}^0)$ is finitely generated for all $k\geqslant p\geqslant 0$. Therefore, since y was chosen arbitrarily, we have proved that Y is clc_R^k. □

3. Discussion

First, we discuss the tightness of our results. Then we contrast the results of this paper against those of [3], by showing that if X is clc, then f is clc, and applying the theorems to shape fibrations. We conclude with a naturally arising question. The coefficient module R in the examples is assumed to be the ring of integers.

Example 3.1. Let $S(n)$ denote the 2-sphere of radius $1/n$ centered at $(1/n, 0, 0)$ in Euclidean 3-space. Let $\Sigma=\bigcup_{n=1}^{\infty}S(n)$. Let f be the identity map on Σ. Then f is ST^2, and clc^1, but the image of f is not clc^2. This shows the clc requirement on f in Theorem 2.2 is necessary. Also, f has finite cohomological type through dimension 2 and its image is clc^1 but f is not clc^2.

Example 3.2. Let H denote the Hawaiian Earring and let f be a constant map on H. Then f is ST^1, does not have finite cohomological type in dimension 1, and the image of f is clc^1. However f itself is not clc^1, showing that the requirement on the cohomological type of f in Theorem 2.1 is necessary; in particular, it shows that the converse of Proposition 1.1 is false.

Example 3.3. Let S^2 denote the standard 2-sphere in Euclidean 3-space. Let $G_n=\{(x, y, z)\in S^2 \mid x=1-1/n\}$. Consider the upper semicontinuous decomposition $\mathscr{G}$ of S^2 whose only nondegenerate elements are the G_n. Then $S^2/\mathscr{G}$ is not clc^2 at the point corresponding to $(1, 0, 0)$; moreover, the decomposition map is clc^2 and also ST^0, but *not* ST^1. This shows that the condition on the sheaf-triviality is necessary in Theorem 2.2.

Example 3.4. Let $S(n)$ be the circle in Euclidean 3-space of radius $1/n$ centered at $(1/n, 0, n-1)$ in the plane parallel to the xy-plane. Let $X_0 = \{(x, y, z) \mid z \geqslant 1, x = y = 0\} \cup (\bigcup_{n=1}^{\infty} S(n))$ and $X = X_0 \cup \{\infty\}$, the one-point compactification of X_0. Let f be the map from X to $[0, 1]$ where $f^{-1}(1 - 1/n) = S(n)$, $f^{-1}(1) = \infty$, and otherwise $f^{-1}(z) = (0, 0, z/(1-z))$. Then f is LST^0 (but *not* LST^1), has finite cohomological type and clc^1 image, yet f is clc^1.

Along with the example in sequel, the following shows that Theorems 2.1 and 2.2 generalize Corollaries 1 and 2 of [3].

Corollary 3.5. *Suppose $f : X \to Y$ is a proper map between locally compact metrizable spaces, X is clc^{n+1} and f is LST^n. Then f is clc^{n+1}, if either*
(1) *Y is clc^{n+1} or*
(2) *$f^{-1}(y)$ has finite cohomological type through dimension n for all $y \in Y$.*

Proof. Under these hypotheses, (1) holds if and only if (2) holds [3]. Then apply Theorem 2.1. □

Example 3.6. Let W be the Warsaw circle and S the topological circle. Let A be the arc of points in $X = W$ at which X is not locally connected. Then let $Y = W/A \cong S$. Let f be the quotient map; note that f is a shape fibration shape-equivalent to the identity map on S. Then f is a shape fibration satisfying the conditions of Theorem 2.1 without X being 0-clc (the Dydak–Walsh result does not apply to this map). Note further that the point pre-images of f are clc; the point being made here is that even if f, Y and point pre-images of f are all clc, and f has finite cohomological type, the local connectivity structure of X may be relatively pathological.

The following determines the necessary condition for a shape fibration with fibers of finite cohomological type to have locally cohomologically connected image.

Corollary 3.7. *Suppose $f : X \to Y$ is a surjective shape fibration between locally compact metrizable spaces and f has finite cohomological type through dimension n. Then Y is clc^{n+1} if and only if f is clc^n.*

Proof. Theorem 3 of [3] shows that f is LST^n. Apply Theorems 2.1 and 2.2. □

Example 3.8. Let W be the Warsaw circle and S the topological circle. Let $X = W \times S$, $Y = W$, and $f : X \to Y$ projection onto the first factor. This example illustrates a shape fibration that is not a clc map yet with fibers having finite cohomological type; thus the image is not clc.

Example 3.2 shows that Y may be clc without either X or f being clc or the fibers of f having finite cohomological type, which leads us to the question: what is a "natural" condition on f which omits any reference to finite-generation of cohomology modules which is equivalent to the image of f being clc?

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