Homotopy properties of decomposition spaces *

by

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Abstract. Let X be a continuum (compact, connected set) in E^n. Then the homotopy type of the decomposition space E^n/X depends only on the shape of X. We also show a necessary and sufficient condition for E^n/X to be locally simply connected. This is the "*nearly*-1-*movable*" property of continua described by D. R. McMillan. Thus the local simple connectedness of decomposition space also depends only on the shape of X.

Introduction. Let X be a continuum (compact, connected set) in Euclidean n-space E^n. We investigate the homotopy properties of decomposition space E^n/X obtained by identifying X to a point and giving the resulting space the quotient topology.

We first show that the homotopy type of E^n/X depends only on the shape of X. This generalizes previous results of D. Henderson [7], S. Mardesič [9], and R. Geoghegan and R. Summerhill [6]. There are continua X and Y which have the same shape, but their decomposition spaces are not homeomorphic (for example, two arcs in E^3, one cellular and one noncellular). On the other hand, there are homeomorphic decomposition spaces of two continua X, Y where X, Y do not have the same shape.

D. R. McMillan [11] defined the concept of "*nearly*-1-*movable*". We show that the property of a continuum being nearly-1-movable is necessary and sufficient for E^n/X being locally simply connected. Thus by [12], this property is also equivalent to E^n/X being simply connected. As a corollary we get the results that E^3 modulo a solenoid or E^3 modulo the 'Case-Chamberlin continuum' [4] are not simply connected. The first result was announced by R. H. Bing in [1]. The second result was shown by S. Armentrout. Both proofs are unpublished.

Throughout the paper we use the geometric approach to Shape theory as defined by Borsuk [2].

§ **1.** We show that if two continua X and Y in E^n have the same shape then their decomposition spaces have the same homotopy type.

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Let $Q = \prod_1^\infty [-1, 1]$ denote the Hilbert cube. We identify $E^n = \prod_1^n (-1, 1)$ in the first n factors of Q, and the unit ball

$$B^n = \prod_1^n [-\tfrac{1}{2}, \tfrac{1}{2}] \subset E^n \subset Q .$$

If X is a continuum in B^n, B^n/X can be considered as a subset of $E^n/X \subset Q/X$. We state a theorem of K. Borsuk [3] in this notation.

THEOREM [K. Borsuk]. *Let W be a strong deformation retract of $\hat{W}$. Let X be a continuum in W. Then W/X is a strong deformation retract of $\hat{W}/X$.*

Thus in particular, $\hat{W}/X$ and W/X have the same homotopy type.

COROLLARY. *Since B^n is a strong deformation retract of both Q and E^n, therefore Q/X and E^n/X have the same homotopy type.*

THEOREM 1. *Let $X, Y \subset E^n$ be continua such that* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. *Then E^n/X has the same homotopy type as E^n/Y.*

Proof. Since we are considering E^n as imbedded in the first n factors of Q, X and Y are z-sets in Q. Thus by Chapman [5],

$$Q - X \text{ is homeomorphic to } Q - Y.$$

Let $h\colon Q - X \to Q - Y$ be a homeomorphism.

Define $\bar{h}\colon Q/X \to Q/Y$ to be $\bar{h}(x) = h(\bar{x})$, $\bar{x} \notin X$, $\bar{h}(\bar{X}) = \bar{Y}$. Then $\bar{h}$ is continuous since h is a proper map. Since $\bar{h}$ is a 1 to 1, continuous function between compact spaces, it is a homeomorphism. Thus $Q/X \cong Q/Y$.

By corollary above E^n/X has the same homotopy type as E^n/Y.

QUESTION. Let X, Y be continua in E^n. Let $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$. Does E^n/X homotopically dominate E^n/Y?

Remark. We know by [12] that if X, Y are continua in E^n (or Q), $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$ and E^n/X is simply connected, then E^n/Y is also simply connected.

§ 2. A compact set $X \subset Q$ is said to be *nearly-1-movable* if for some (and hence for every) embedding of X in Q, and each open set U in Q containing X, there is an open set V containing X such that V nearly-1-moves towards X in U.

That is, given any loop

$$l\colon S^1 \to V,$$

and any open W containing X, there is a map

$$g\colon B^2 - \bigcup_{i=1}^n D_i \to U$$

(D_i closed 2-cell $\subset \mathrm{Int}\,B^2$, $i = 1, 2, \ldots, n$, $D_i \cap D_j = \emptyset$, $i \neq j$) such that

$$g|_{\partial B^2} = l \quad \text{and} \quad g(\bigcup \partial D_i) \subset W.$$

In other words, every loop in V belongs to the normal closure in U of every neighborhood W of X.

D.R. McMillan has shown [11] that 1-movability implies near-1-movability and that this implication is irreversible. The solenoids as also the 'Case-Chamberlin continuum' [4] are not nearly-1-movable.

We show first that near-1-movability is a shape property.

LEMMA 2.1. *Let X, Y be continua in Q. If X is nearly-1-movable and* $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$ *then Y is nearly-1-movable.*

Proof. There are fundamental sequences

$$\underline{f} = \{f_k, X, Y\} \quad \text{and} \quad \underline{g} = \{g_k, X, Y\}$$

such that $\underline{f} \circ \underline{g} = \underline{\mathrm{id}}\,Y$.

Let U be any open set containing Y. Then there is

(i) U^1 containing Y and integer $N_1 > 0$ such that

$$f_k \circ g_k|_{U^1} \simeq f_{k+1} \circ g_{k+1}|_{U^1} \simeq \mathrm{id}_Y|_{U^1} \text{ in } U \quad \text{for all } k > N_1 .$$

(ii) There is a U_1 containing X and $N_2 > 0$ such that

$$f_k|_{U_1} \simeq f_{k+1}|_{U_1} \text{ in } U^1 \quad \text{for all } k > N_2 .$$

(iii) There is a V_1 containing X and $N_3 > 0$ such that $X \subset V_1 \subset U$ and V_1 nearly-1-moves towards X in U.

(iv) There is V containing Y and $N_4 > 0$ such that

$$g_k|_V \simeq g_{k+1}|_V \text{ in } V_1 \quad \text{for all } k > N_4 .$$

Then it is easy to see that V nearly-1-moves towards Y in U.

Thus near-1-movability is a shape property. To prove the 'if' part of our main theorem we use the notion of *local-1-connection*, as defined by G. Kozlowski in [8].

DEFINITION. The projection $p\colon E^n \to E^n/X$ is said to be a *local-1-connection* if for each open set U in E^n/X containing $\bar{X} = p(X)$, there is an open V in E^n/X, $\bar{X} \subset V \subset U$ such that every loop in $p^{-1}(V)$ projects to a loop that is homotopic to a constant in U.

THEOREM 2.2. *Let $X \subset E^n$ be a continuum. Then X is nearly-1-movable if and only if E^n/X is locally simply connected.*

Proof. First we show that if X is nearly-1-movable then $p\colon E^n \to E^n/X$ is a local-1-connection.

Let U be an open set containing $p(X) = \bar{X}$. $p^{-1}(U)$ is an open set in E^n and contains X. Since X is nearly-1-movable, there is a sequence of open sets $\{V_i\}_{i=0}^\infty$ with the following properties

(i) $V_0 = p^{-1}(U)$, $X \subset V_i$ for $i = 0, 1, 2, \ldots$,

(ii) $\bar{V}_i \subset V_{i-1}$,

(iii) every loop in V_i nearly-1-moves towards X in V_{i-1}.

We let $V = p(V_i)$, an open set. Let $l\colon S^1 \to V$ be a loop. There is $D_1\colon B^1 - \bigcup_{i=1}^{n_1} B_1^i \to U$ where each B_1^i is a 2-cell,

$$B_1^i \cap B_1^j = \emptyset, \quad i \neq j,$$
$$\bigcup B_1^i \subset \operatorname{int} B^1 \quad \text{and} \quad \operatorname{diam}(B_1^i) < 1,$$

such that

$$D_1|_{\partial B^1} = l \quad \text{and} \quad D_1(\bigcup \partial B_1^i) \subset V_2, \quad i = 1, 2, \ldots, n.$$

Now $D|_{\partial B_1^i}$ is a loop in V_2 so there is $D_2^i\colon B_1^i - \bigcup_{j=1}^{m(2,i)} B_2^j \to V_1$ such that $\operatorname{diam} B_2^j < \frac{1}{2}$ and $D_2^i|_{\partial B_1^i} = D_1|_{\partial B_1^i}$, and

$$D_2^i(\bigcup \partial B_2^j) \subset V_3.$$

We continue in this manner. Since the union of the ith stage is contained in some 2-cell B_j^{i-1} at the $(i-1)$-st stage, it is possible to get a map D of B^1 minus a zero dimensional set S. (This is possible since the diameter of each B_i^j is less than $1/j$.)

We define a map from B^1 to U as follows:

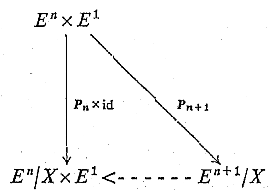
$$D^1(y) = \begin{cases} P \circ D(y), & \text{if } y \in B^1 - S, \\ p(X), & \text{if } y \in S. \end{cases}$$

Since the image under D_j^i of the union of ∂B_j^i at each stage is contained in V_{i+1}, and the image of the zero dimensional set under D is contained in $p(X)$, therefore D^1 is continuous. Thus $D^1\colon B^1 \to U$ extends $p \circ l\colon S^1 \to P(V_1) = V$. Hence every loop in V_1 projects to a loop which homotopes to a constant in U. Thus p is a local-1-connection. To show that this implies that E^n/X is locally simply connected, we can apply lemma [1] of G. Kozlowski [8].

Conversely, assume E^n/X is locally simply connected. Let $X \subset E^n$, $X\{0\} \subset E^{n+1}$.

We consider X as a subset of $E^n \times \{0\}$ embedded in E^{n+1} as shown above. We work in E^{n+1} to find sufficient space to shrink loops.

Consider this diagram



Define $F\colon E^{n+1}/X \to E^n/X \times E^1$ to be $F(\overline{y, t}) = (\bar{y}, t)$. Then $F \circ P_{n+1} = P_n \times \mathrm{id}$.

It is easy to show that F is well defined and continuous. Let U be an open set in E^{n+1}/X containing $\bar{X} = P_{n+1}(X)$.

$P_{n+1}^{-1}(U)$ is an open set in E^{n+1} and contains X. Let $U^1 = P_{n+1}^{-1}(U) \cap E^n$ which is open in E^n; and $X \subset U^1$.

Since E^n/X is locally simply connected, there is an open set V^1 in $E^n \times \{0\}$ with $X \subset V^1 \subset U^1$ such that every loop in $P_n(V^1)$ shrinks in $P_n(U^1)$.

There is an $\varepsilon > 0$ such that $V^1 \times [-\varepsilon, \varepsilon] \subset P_{n+1}^{-1}(U)$. Let $V = P_{n+1}(V^1 \times (-\varepsilon, \varepsilon))$; which is contained in U. We want to show that each loop in V shrinks in U. It is sufficient to show that P_{n+1} is a local 1-connection.

Let $l\colon S^1 \to V^1 \times (-\varepsilon, \varepsilon)$. Then l is freely homotopic in $V^1 \times (-\varepsilon, \varepsilon)$ to a loop l^1 in $V^1 \times \{0\}$. Now

$$P_n(l(S^1)) \subset P_n(V^1 \times \{0\}) \subset E^n/X \times \{0\}.$$

Hence $P_n \circ l\colon S^1 \to P_n(V^1 \times \{0\})$ extends to $g\colon B^2 \to P_n(U^1)$, so $F^{-1} \circ g|_{\partial B^2} = P_{n+1} \circ l|_{\partial B^2}$.

Thus $P_{n+1} \circ l(S^1)$ shrinks in $P(U) \subset E^{n+1}/X$. Therefore E^{n+1}/X is locally simply connected.

Now we show that X is nearly-1-movable as a subset of E^{n+1}. Let U be open in E^{n+1} containing X. Choose $V^1 \subset P(U)$ by local simple connectedness. Let $V = P^{-1}(V^1)$. Let $l\colon S^1 \to V$ be a loop and let W, $X \subset W \subset V$, W open, be given. We have to show that l belongs to the normal closure in U of W.

We can assume that $p \circ l(S^1)$ misses $P(X)$. For l is homotopic in V to a loop that misses X. $P \circ l\colon S^1 \to P(V) = V^1$ extends to a map $g\colon B^2 \to P(U)$.

Consider $g^{-1}(P(X))$ which is a compact set in the interior of B^2.

Then $g^{-1}(P(X)) \subset g^{-1}(P(W)) \subset B^2$.

We can find a finite number of disjoint simple closed curves $R_1, R_2, \ldots, R_n$ with the following properties.

Let B_i denote the component of $B^2 - R_i$ that misses ∂B^2. Then the B_i's are disjoint and $\bigcup B_i$ contains $g^{-1}(P(X))$ and such that the images of these simple closed curves R_i lie in W. (Such a collection of simple closed curves can be obtained by taking a brick decomposition of B^2 that has mesh smaller than

$$\tfrac{1}{10}\left(\operatorname{dist}\left(g^{-1}(P(X)), B^2 - g^{-1}(P(W))\right)\right)$$

and taking the relevant part of the boundary of the star of $g^{-1}(P(X))$.)

Now $g(B^2 - \bigcup_{i=1}^{n} B_i)$ can be lifted to U. Thus there is a map

$$p^{-1} \circ g = \bar{g}\colon B^2 - \bigcup_{i=1}^{n} B_i \to U$$

such that

$$\bar{g}(\partial B_i) \subset W, \quad \bar{g}|_{\partial B^2} = l.$$

So l belongs to the normal closure in U of W. Therefore X is nearly-1-movable.

§ 3. Movability properties are related to the UV properties [10] as follows. Property 1-UV for a compactum X clearly implies 1-movability.

Conversely,

THEOREM 3.1. *Let X be a continuum in E^n having the property that for any neighborhood U of X the only loop that belongs to the normal closure in U of each neighborhood W of X is the trivial loop. Then X is nearly-1-movable if and only if X is* 1-UV.

Proof. Let X be nearly-1-movable. Let U be an open set containing X. Choose V so that each loop in V belongs to the normal closure in U of each open W. $X \subset W \subset V$. But only such loops are trivial loops. Thus X is 1-UV.

COROLLARY. *If X is as above, then X has property* 1-UV *if and only if E^n/X is locally simply connected.*

Proof. Clear.

As a corollary, we get the following theorem of D. R. McMillan [10].

THEOREM. *If X is compact connected strongly 1-acyclic, then X is* 1-UV *if and only if E^n/X is locally simply connected.*

Proof. Strongly acyclic continua satisfy the property in Theorem 3.1.

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Yosida-Fukamiya's theorem for f-rings

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Abstract. We introduce the concept of super-infinitely small element and prove that in a commutative f-ring with unity the J-radical coincides with the set of all super-infinitely small elements.

Preliminaries. We follow the notation and terminology of [1] and [5]. A lattice-ordered ring is an *f-ring* if $ax \wedge y = xa \wedge y = 0$ whenever $x \wedge y = 0$ and $a \geqslant 0$. If we put $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ + x^-$, then a lattice-ordered ring is a *d-ring* if $|xy| = |x| \cdot |y|$, $\forall x, y$. The term ideal must be understood in the ring-theoretic sense. An ideal I is an *l-ideal* if $|x| \leqslant |y|$, $y \in I \Rightarrow x \in I$. We denote by $\langle a \rangle$ the l-ideal generated by $a \in A$. Following [1], an element $a \in A$ such that $\langle a \rangle = A$ is called a *formal unity*. An l-ideal I is a *band* if, whenever a subset of I has a supremum in A, that supremum belongs to I. The *J-radical* $J(A)$ of an f-ring A is defined as the intersection of all maximal (two-sided) l-ideals, if there is any. Otherwise, $J(A) = A$ by definition. The ring A is *J-semisimple* if $J(A) = 0$. An element $x \in A$ is *infinitely small* with respect to the element $y \in A$ whenever $n|x| \leqslant |y|$ holds for $n = 1, 2, \ldots$ If we put $I_0(A) = \bigcup_{y \in A} I_0(y)$, where $I_0(y) = \{x \in A \mid x$ is infinitely small with respect to $y\}$, then A is Archimedean if and only if $I_0(A) = 0$. A lattice-ordered ring is *Dedekind complete* if every non-empty subset which is bounded from above has a supremum.

Introduction. In vector lattices with a strong unit the Yosida–Fukamiya's theorem [7] asserts that the *radical* — intersection of all maximal l-vector subspaces — is the set of all infinitely small elements. Here, for a commutative f-ring with unity, we obtain a result that is parallel to that of Yosida–Fukamiya. But in this context infinitely small elements are no more appropriate and it has been necessary to introduce a notion of "smallness" related to the product of the ring: that of *super-infinitely small element*. And the set of all super-infinitely small elements of A is proved to be $J(A)$.

Super-infinitely small elements and pseudoarchimedean rings.

DEFINITION 1. The element x of the lattice-ordered ring A is called *super-infinitely* small element with respect to $y \in A$ whenever $|a| \cdot |x| \leqslant |y|$ and $|x| \cdot |a| \leqslant |y|$ hold for every $a \in A$.