A Vietoris-Begle Theorem for Differentiable Maps

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The natural approach to a Vietoris–Begle theorem is to consider the Leray spectral sequence for the map. In order to identify the stalks of the Leray sheaf as cohomology modules of the fibres an assumption about the map is necessary. The usual assumption is for the map to be closed. With this assumption, Skljarenko [9] obtained a Vietoris–Begle theorem which allowed exceptional fibres over sets of small relative covering dimension. Working jointly and not using the Leray sequence, Professor Smith and myself [7] obtained a Vietoris–Begle theorem by supposing the map was a submersion (smooth surjection with no critical points). By combining these two ideas, this paper results in a Vietoris–Begle theorem in singular homology for a smooth surjective map with critical values on sets of small relative covering dimension.

Leray Spectral Sequence. If $f:X \to Y$ is a continuous map and if $\mathcal{A}$ is a sheaf on X, denote the Leray sheaf of f with compact support by $\mathcal{H}_c^*(f; \mathcal{A})$. For other basic notation and terminology of sheaf theory, I refer to Bredon's text [1]. By restricting the range and domain of the map to finite dimensional manifolds the Leray spectral sequence can be given in terms of sheaf cohomology with compact support.

Theorem 1. *Let $f:X \to Y$ be a continuous map between finite dimensional manifolds, then for any sheaf $\mathcal{A}$ on X there is a spectral sequence where*

$$E_2^{p,q} = H_c^p(Y; \mathcal{H}_c^q(f; \mathcal{A})) \Rightarrow H_c^{p+q}(X; \mathcal{A})$$

with natural isomorphisms $\mathcal{H}_c^*(f; \mathcal{A})y \approx H_c^*(f_y^{-1}; \mathcal{A})$.

Proof. Since X and Y are finite dimensional they are locally compact. Thus the family of compact supports c is paracompactifying [1, pg 16]. Because Y is Hausdorff and c is paracompactifying, the fibres are c-taut in X [1, pg 52]. Noting that the family of compact supports on the Hausdorff manifolds X and Y are well adapted [1, pg 140], the result now follows from Theorem 4.6.1 of Bredon's text [1].

Dimension Theory. The covering dimension for a normal space X, denoted

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by dim X, is at most n if every finite open covering of X can be refined by an open covering of order at most $n + 1$. If dim $X \leqq n$ and the statement dim $X \leqq n - 1$ is false then we set dim $X = n$. In this paper I will define the covering dimension of the empty set to be $-\infty$. By combining a result of Dieudonné [5, pg 12] with a corollary to a theorem of Morita and Dowker [5, pg 56], we obtain a characterization of the covering dimension for a paracompact Hausdorff space.

Theorem 2. *A paracompact Hausdorff space X has dim $X \leqq n$ if and only if every open covering is refined by an open covering of order less than or equal to $n + 1$.*

If A is a subset of a normal space X, then define the relative covering dimension of A with respect to X by

$$\text{Rel-dim}_X A = \sup \{\dim C: C \text{ is a closed subset of } X \text{ contained in } A\}.$$

From this definition it follows for any subset A of X that $\text{Rel-dim}_X A \leqq \dim X$.

The following lemma was first proved by Skljarenko [8] for sheaf cohomology without a family of supports.

Theorem 3. *Let $\mathcal{A}$ be a sheaf over a Hausdorff space X which is concentrated on a subset A of relative covering dimension d. Let ϕ be any paracompactifying family of supports on X. If $p > d$ then $H^p_\phi(X; \mathcal{A}) = 0$.*

Proof. Because ϕ is paracompactifying there is a canonical isomorphism between Čech and sheaf cohomology with supports ϕ [2, pg 228]. Let $\alpha = \{0_\lambda : \lambda \in \Lambda\}$ be any open cover of X and let z_α be an arbitrary cocycle in the nerve of α representing an element in $\check{H}^p_\phi(X; \mathcal{A})$. Since ϕ is paracompactifying there is a closed paracompact set P such that $|z| \subset \text{int } P \subset P$. Because P is paracompact the cover α restricted to P has an open locally finite refinement $\beta = \{U_\lambda : \lambda \in \Lambda\}$. Again, since P is paracompact β has an open locally finite refinement $\gamma = \{V_\lambda : \lambda \in \Lambda\}$ such that $\bar{V}_\lambda \subset U_\lambda$ for all $\lambda \in \Lambda$.

Let $K_{\lambda_0 \cdots \lambda_p}$ be an arbitrary simplex in the nerve of α on which z_α is nonvanishing. Clearly $K_{\lambda_0 \cdots \lambda_p} \subset A \cap \text{int } P$. Now define $L_{\lambda_0 \cdots \lambda_p} = \bigcap_{i=0}^p \bar{V}_{\lambda_i}$. Then $L_{\lambda_0 \cdots \lambda_p}$ is a closed subset of P contained in $A \cap \text{int } P$, and hence is closed in X. Let I be the set of indices of all p-simplexes in the nerve of α on which z_α is nonvanishing. Since the cover β is locally finite, the collection of subsets $\{L_{\lambda_0' \cdots \lambda'_p} : \lambda_0' \cdots \lambda_p' \in I\}$ is locally finite. Thus $L = \bigcup_I L_{\lambda_0 \cdots \lambda_p}$ is a closed subset of X contained in P and hence paracompact.

Since L is a closed subset of X contained in A and $\text{Rel-dim}_X A = d$, the covering dimension of L is at most d. Because L is a paracompact Hausdorff space, by Theorem 2 there is an open cover $\delta = \{W_\lambda : \lambda \in \Lambda\}$ of L having order at most d which refines the open cover γ restricted to L. Consider the refinement η of the original cover α given by

$$\eta = \{X_\lambda : X_\lambda = (0_\lambda - L) \cup (W_\lambda \cap \text{int } L) \quad \text{for all } \lambda \in \Lambda\}.$$

Clearly η is an open cover of X. Let z_η be the restriction of z_α to the refinement η.

By the construction the space of any p-simplex in the nerve of η lies in the complement of the support of z_η. Consequently the Čech cohomology is trivial.

An immediate consequence of Theorem 3 is the following corollary.

Corollary 4. *Let A be a closed subset of a Hausdorff space X with dim $A = d$. Let ϕ be any paracompactifying family of supports and $\mathfrak{A}$ any sheaf on X. If $i > d + 1$ then $H^i_{\phi|X-A}(X - A; \mathfrak{A}) \approx H^i_\phi(X; \mathfrak{A})$.*

Proof. Since dim $A = d$ by Theorem 3 $H^i_{\phi|A}(A; \mathfrak{A}) = 0$ for all $i > d$. Applying this fact to the standard long exact sequence [2, pg 190], the result follows.

The last dimension theory result needed in this paper is a reformulation of Sard's Theorem in terms of the covering dimension of the set of critical values.

Theorem 5. *The set of critical values of a C^∞ map $f:N \to M$ between manifolds of dimension n and m respectively has covering dimension less than m.*

Proof. By Sard's Theorem the Lebesgue outer measure of the set of critical values in zero. Thus the m-measure of this set is trivial. [3, pg 104]. Which implies that the covering dimension is less than m. [3, Thm. 7.2 pg 104].

Vietoris–Begle Theorem. Let $f:N \to M$ be a C^∞ map between manifolds of dimension n and m, respectively. Let G be any finitely generated module over a principal ideal domain, and define $\mathcal{O}_N$ to be the orientation sheaf on N with respect to G. Denote the Borel–Moore homology modules of X with coefficient sheaf $\mathfrak{A}$ and family of supports ϕ by $\bar{H}^\phi_*(X; \mathfrak{A})$. For the basic theorems in Borel–Moore homology theory I will refer to chapter five of Bredon's text [1].

Applying Theorem 1 to the map $f:N \to M$ and the orientation sheaf $\mathcal{O}_N$ on N, the stalk of the Leray sheaf is $H^*_c(f_y^{-1}; \mathcal{O}_N)$. The following lemma identifies this stalk with a relative singular homology module.

Lemma 6. *There is a natural isomorphism*

$$H^p_c(f_y^{-1}; \mathcal{O}_N) \approx H_{n-p}(N, N - f_y^{-1}; G).$$

Proof. Since f_y^{-1} is a closed subset of N, the basic spectral sequence in Borel–Moore homology gives a natural isomorphism $H^p_c(f_y^{-1}; \mathcal{O}_N) \approx \bar{H}^c_{n-p}(N, N - f_y^{-1}; G)$ [1, pg 208]. Since N and $N - f_y^{-1}$ are both n-dimensional manifolds, which are clearly HLC, then there is a natural isomorphism

$$\bar{H}^c_{n-p}(N, N - f_y^{-1}; G) \approx H_{n-p}(N, N - f_y^{-1}; G) \qquad [1, \text{pg } 231].$$

Because the fibre of f over a regular value is a submanifold which admits a tubular neighborhood, the stalk over this value can be identified with the homology of its fibre.

Lemma 7. *If y is a regular value of f then*

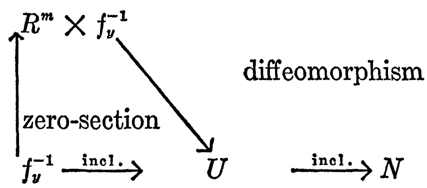
$$H^p_c(f_y^{-1}; \mathcal{O}_N) \approx H_{n-m-p}(f_y^{-1}; G).$$

Proof. Since y is a regular point, f_y^{-1} is a submanifold of N. Thus f_y^{-1} is paracompact and hence admits smooth partitions of unity [4, pg. 30].

Because df is surjective over regular points, the following tangent bundle sequence is exact.

$$0 \to Tf_y^{-1} \to TN|_{f_y^{-1}} \xrightarrow{df} TM|_y \to 0.$$

Since f_y^{-1} is paracompact, $TN|_{f_y}^{-1}$ admits a Riemannian metric and the exact sequence splits. Hence the induced bundle $f^*(TM|_y) \approx R^m \times f_y^{-1}$ is a normal bundle for f_y^{-1}. Because f_y^{-1} admits smooth partitions of unity and $f^*(TM|_y)$ is a Hilbert bundle, $f^*(TM|_y)$ is compressible [4, pg 106]. Thus the standard tubular neighborhood construction [3, pg 73–75] gives the following diagram where U is an open subset of N.



By excising the complement of U and noting that U is diffeomorphic to $R^m \times f_y^{-1}$ with f_y^{-1} going to the 0-section, we obtain

$$H_{n-p}(N, N - f_y^{-1}; G) \approx H_{n-p}(R^m \times f_y^{-1}, R^m - \{0\} \times f_y^{-1}; G).$$

But by the relative Künneth formula,

$$H_{n-p}(N, N - f_y^{-1}; G) \approx H_{n-p-m}(f_y^{-1}; G).$$

By composing this isomorphism with that in Lemma 6, the result follows.

In the special case, when f is a submersion with connected fibres, the Leray sheaf of dimension $n - m$ is just the orientation sheaf on M.

Lemma 8. *If $f:N \to M$ is a submersion with connected fibres then f induces an isomorphism*

$$\mathcal{H}_c^{n-m}(f; \mathcal{O}_N) \approx \mathcal{O}_M .$$

Proof. Consider the stalk of the sheaf $\mathcal{H}_c^{n-m}(f; \mathcal{O}_N)$ over y. From the definition of the Leray sheaf we have

$$\mathcal{H}_c^{n-m}(f; \mathcal{O}_N)_y = \lim_{y \in U} H^{n-m}_{c \cap f^{-1}U}(f^{-1}U; \mathcal{O}_N)$$

and from the definition of $\mathcal{O}_M$ we have

$$\mathcal{O}_{M|y} = \lim_{y \in U} H_m(M, M - U; G).$$

For every open set U contained in a disk D in M we obtain

$$H^{n-m}_{c \cap f^{-1}U}(f^{-1}U; \mathcal{O}_N) \overset{\Delta}{\approx} H_m(f^{-1}D, f^{-1}D - f^{-1}U; G) \xrightarrow{f_*} H_m(D, D - U; G)$$

with the first isomorphism being the natural map given by generalized Poincaré duality [1, pg 209]. Because of the naturality of the map $f^* \circ \Delta$, there is a map induced on the presheaves, which in turn induces a continuous map $\bar{f}$ between the sheaves [1, pg 7].

$$\bar{f}: \mathcal{H}_c^{n-m}(f; \mathcal{O}_N) \to \mathcal{O}_M .$$

Since f is a submersion, there exist local cross-sections. Hence for small disks in M, f^* has a right inverse, thus $\bar{f}$ is surjective. Because the fibres are connected, the stalks of the Leray sheaf, by Lemma 7, are just the module G. Hence $\bar{f}$ restricted to a stalk is a homomorphism of G onto itself. Since G is a finitely generated module over a *PID*, $\bar{f}$ must be injective. Hence $\bar{f}^{-1}$ exists stalkwise. But the local cross-section maps induce maps between the directed systems. Since these maps induce $\bar{f}^{-1}$ stalkwise, $\bar{f}^{-1}$ must be continuous.

Because we will require that the regular fibres have trivial homology in some range of dimensions, the exceptional fibres will occur only over certain critical values. Let CV be the set of critical values of f in M and $\overline{CV}$ be its closure. Now define

$$T_i = \{y \in M : y \in CV \quad \text{and} \quad H_i(N, N - f_y^{-1}; G) \neq 0\}$$

for all $i \neq m$. When $i = m$, let T_m be any open set in the relative topology of $\overline{CV}$ which contains

$$\{y \in M : y \in CV \quad \text{and} \quad H_m(N, N - f_y^{-1}; G) \neq 0\}.$$

Clearly $\overline{CV} - T_m$ is a closed subset of M. Define the open submanifold M' by $M' = M - \overline{CV}$.

Vietoris-Begle Theorem. *Let $f: N \to M$ be a surjective differentiable map between manifolds of dimension n and m. Suppose there is an integer t such that all the regular fibres are homology acyclic in dimensions $[t - m, t]$.*

Case I. $t > m$
If $i >$ Rel-dim$_M$ T_{t+i} for $0 \leqq i \leqq m - 1$, then $H_t(N; G) = 0$.
Case II. $t \leqq m$
If $i - 1 >$ Rel-dim$_M$ T_{t+i} for $-1 \leqq i \leqq m - t$ and if $i >$ Rel-dim$_M$ T_{t+i} for $m - t + 1 \leqq i \leqq m - 1$, then there is an isomorphism $H_t(N; G) \approx H_t(M'; G)$ and a monomorphism $H_{t-1}(N; G) \to H_{t-1}(M'; G)$.

In the case of a submersion, we obtain the following corollary.

Corollary. *Let $f: N \to M$ be a submersion. Suppose there is an integer t such that all the fibres are homology acyclic in dimensions $[t - m, t]$. If $t > m$ then $H_t(N; G) = 0$. If $t \leqq m$ then the induced map on homology $f_*: H_i(N; G) \to H_i(M; G)$ is an isomorphism for all $i \leqq t$.*

Proof of Vietoris–Begle Theorem. Suppose y is a regular value, then by the hypothesis $H_k(f_y^{-1}; G) = 0$ for $k \in [t - m, t]$ with $k \neq 0$. Thus by Lemma 7, the stalk of the Leray sheaf $\mathcal{H}_c^j(f; \mathcal{O}_N)_y$ is trivial for $j \in [n - m - t, n - t]$ with

$j \neq n - m$. Hence by Lemma 6, the Leray sheaf is concentrated over the set T_{n-j}. Now consider the E_2 terms of the Leray spectral sequence given by Theorem 1.

$$E_2^{i,n-t-i} = H_c^i(M; \mathcal{H}_c^{n-t-i}(f; \mathcal{O}_N))$$

Since $i >$ Rel-dim$_M$ T_{t+i} for $0 \leqq i \leqq m - 1$ and $\mathcal{H}_c^{n-t-i}(f; \mathcal{O}_N)$ is concentrated on T_{t+i} except for $i = m - t$, Theorem 3 implies that $E_2^{i,n-t-i} = 0$ for $0 \leqq i \leqq m - 1$ except for $i = m - t$. Combining Theorem 5 with Theorem 3 we obtain $E_2^{i,n-t-i} = 0$ for $i = m$ (except when $t = 0$). Because the dimension of M is m, $E_2^{i,n-t-i} = 0$ for all $i < 0$ and for all $i > m$. Thus $E_2^{i,n-t-i} = 0$ for all i except when $t \leqq m$ and $i = m - t$.

Case I. Since $t > m$, all the E_2 terms of total degree $n - t$ are trivial. Thus by the standard spectral sequence argument [6, pg 435], $H_c^{n-t}(N; \mathcal{O}_N) = 0$. By applying Poincaré duality and the natural isomorphism of Borel–Moore homology with singular homology [1, pg 225], we obtain $H_t(N; G) = 0$.

Case II. By the above argument there is only one nontrivial E_2 term of total degree $n - t$, namely

$$E_2^{m-t,n-m} = H_c^{m-t}(M; \mathcal{H}^{n-m}(f; \mathcal{O}_N)).$$

In the same manner the hypotheses imply there is only one non-trivial E_2 term of total degree $n - t + 1$, namely

$$E_2^{m-t+1,n-m} = H_c^{m-t+1}(M; \mathcal{H}^{n-m}(f; \mathcal{O}_N)),$$

and that all the E_2 terms of total degree $n - t - 1$ and filtering degree less than $m - t - 1$ are trivial. Since the dimension of M is m, all the E_2 terms of filtering degree greater than m are trivial. Thus Serre's Theorem [6, pg 435] can be applied, giving the following exact sequence.

$$0 \to H_c^{n-t}(N; \mathcal{O}_N) \to H_c^{m-t}(M; \mathcal{H}^{n-m}(f; \mathcal{O}_N)) \to 0 \to H_c^{n-t+1}(N; \mathcal{O}_N) \to H_c^{m-t+1}(M; \mathcal{H}^{n-m}(f; \mathcal{O}_N)).$$

From the definition of T_m the set $\overline{CV} - T_m$ is closed. Hence its complement $A = \sim\overline{CV} \cup T_m$ is locally closed. Since $\mathcal{H}^{n-m}(f; \mathcal{O}_N)$ is concentrated on A, $H_c^i(M; \mathcal{H}^{n-m}(f; \mathcal{O}_N)) \approx H_c^i(A; \mathcal{H}^{n-m}(f; \mathcal{O}_N)|_A)$ [1 pg 51]. Since $T_m \subset \overline{CV}$, it is a closed subset of A. Because $m - t - 1 >$ Rel-dim$_M$ T_m, Corollary 4 gives an isomorphism

$$H_c^i(M'; \mathcal{H}^{n-m}(f; \mathcal{O}_N)) \approx H_c^i(A; \mathcal{H}^{n-m}(f; \mathcal{O}_N)|_A) \quad \text{for all} \quad i \geqq m - t.$$

But on M', $\mathcal{H}^{n-m}(f; \mathcal{O}_N)$ is just the orientation sheaf, by Lemma 8. Applying Poincaré duality and the natural isomorphism of Borel–Moore homology with singular homology, we obtain

$$H_c^i(M; \mathcal{H}^{n-m}(f; \mathcal{O}_N)) \approx H_{m-i}(M'; G) \quad \text{for all} \quad i \geqq m - t.$$

Combining these isomorphisms with the above exact sequence, the result follows.

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