FACTORIZATION OF CERTAIN MAPS UP TO HOMOTOPY

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If $f: X \to Y$ is a map of a space X into a space Y, we say that f is a *local connection in dimension n*, provided that for every point $y \in Y$ and every neighborhood N of y there is a neighborhood $V \subset N$ of y such that for $0 \leqq k \leqq n-1$ any map $g: S^k \to f^{-1}V$ extends to a map $g': B^{k+1} \to f^{-1}N$ and for any map $g: S^n \to f^{-1}V$ the map $fg: S^n \to V$ extends to a map $h: B^{n+1} \to N$. Using star-refinements of open covers and a standard approximation technique we establish the following theorem (a slightly weaker form of which has been announced by Price [2]).

THEOREM 1. *Let Y be a metric space, and let $f: X \to Y$ be a local connection in dimension n with dense image. Let L be a subcomplex of a finite simplicial complex K such that* $\dim(K-L) \leqq n$, *and let $g: L \to X$ and $h: K \to Y$ be maps such that $h|L = fg$. Then there is a map $g': K \to X$ such that $g'|L = g$ and fg' is homotopic to h relative to L. If d is any metric for Y and* $\epsilon > 0$, *the map g' and the homotopy H may be chosen so that for all points $p \in K$ the diameter (with respect to d) of $H(p \times I)$ is* $< \epsilon$.

This implies that f is an n-equivalence; i.e. f maps the set of path-components of X bijectively to the set of path-components of Y, and that for every $x \in X$, $f_{\#}: \pi_k(X, x) \subset \pi_k(Y, f(x))$ is an isomorphism for $1 \leqq k \leqq n-1$ and an epimorphism for $k = n$. Since $f|f^{-1}W$ is also a local connection in dimension n for every open set $W \subset Y$, it follows that Y is LC^n. Using these facts and the lemmas for the proof of Theorem 1 we obtain sharper forms of known results:

THEOREM 2 (CF. SMALE [3]). *Let X be a paracompact* LC^n *space, let Y be a metric space, and let $f: X \to Y$ be a closed map of X onto Y such that $f^{-1}(y)$ is* LC^{n-1} *and $(n-1)$-connected for every $y \in Y$. Then Y is* LC^n, *and f is an n-equivalence.*

THEOREM 3 (CF. KWUN [1]). *Let M be a manifold, and let G be an upper semicontinuous decomposition of M into cellular sets. If the*

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decomposition space M/G is finite dimensional, it is a homotopy manifold.

1. All complexes will be finite simplicial complexes, and (K, L) will be called an n-pair provided that L is a subcomplex of K and $\dim (K-L) \leqq n$. The q-skeleton of K will be denoted K^q. If $\mathfrak{U}$ is a collection of open sets in Y, then a map $h: K \to Y$ (a homotopy $H: K \times I \to Y$) will be said to *map* K (resp. $K \times I$) *into* $\mathfrak{U}$ provided that for every (closed) simplex α of K there is $U \in \mathfrak{U}$ with $h(\alpha) \subset U$ (resp. $H(\alpha \times I) \subset U$). Associated with $\mathfrak{U}$ is the collection $\mathfrak{U}^* = \{ U^* \mid U \in \mathfrak{U} \}$, where $U^* = \bigcup \{ U' \in \mathfrak{U} \mid U \cap U' \neq \varnothing \}$. A map $f: X \to Y$ will be said to be a *strong local connection in dimension n*, if for every point $y \in Y$ and every neighborhood N of y there is a neighborhood $V \subset N$ of y such that for $0 \leqq k \leqq n$ any map $g: S^k \to f^{-1}V$ extends to a map $g': B^{k+1} \to f^{-1}N$.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be open covers of a space Y such that $\mathfrak{V}$ refines $\mathfrak{U}$, and let $f: X \to Y$ be a map. For every nonnegative integer n we define assertions $E(\mathfrak{V}, \mathfrak{U}; n)$, $H(\mathfrak{V}, \mathfrak{U}; n)$, and $H(\mathfrak{V}, \mathfrak{U}; f; n)$ as follows:

$E(\mathfrak{V}, \mathfrak{U}; n)$. If (K, L) is any n-pair and $g: L \to X$, $h: K \to Y$ are any maps such that h extends fg and maps $\mathrm{cl}(K-L)$ into the collection $\{ V \in \mathfrak{V} \mid f(X) \cap V \neq \varnothing \}$, then there is an extension $g': K \to X$ of g such that for every simplex α of $\mathrm{cl}(K-L)$ there is $U \in \mathfrak{U}$ with $fg'(\alpha) \cup h(\alpha) \subset U$.

$H(\mathfrak{V}, \mathfrak{U}; n)$. If (K, L) is any n-pair and $g: L \to X$, $g': K \to X$, $g'': K \to X$ are any maps such that $g = g' \mid L = g'' \mid L$ and for every simplex α of K there is $V \in \mathfrak{V}$ with $g'(\alpha) \cup g''(\alpha) \subset f^{-1}V$, then there is a homotopy $G: g' \simeq g''$ relative to L which maps $K \times I$ into $f^{-1}\mathfrak{U} = \{ f^{-1}U \mid U \in \mathfrak{U} \}$.

$H(\mathfrak{V}, \mathfrak{U}; f; n)$. If (K, L) is any n-pair and $g: L \to X$, $g': K \to X$, $g'': K \to X$ are any maps such that $g = g' \mid L = g'' \mid L$ and for every simplex α of K there is $V \in \mathfrak{V}$ with $g'(\alpha) \cup g''(\alpha) \subset f^{-1}V$, then there is a homotopy $H: fg' \simeq fg''$ relative to L which maps $K \times I$ into $\mathfrak{U}$.

LEMMA 1. *Let Y be paracompact, and let $f: X \to Y$ be a strong local connection in dimension n. Then for any open cover $\mathfrak{U}$ of Y there is an open cover $\mathfrak{V}$ of Y refining $\mathfrak{U}$ such that both $E(\mathfrak{V}, \mathfrak{U}; n+1)$ and $H(\mathfrak{V}, \mathfrak{U}; n)$ hold.*

PROOF. For $n = -1$ there are no conditions on the map f, and both assertions hold for $\mathfrak{V} = \mathfrak{U}$. Assume that the lemma is true for $n < k$, and let $f: X \to Y$ be a strong local connection in dimension k. If $\mathfrak{U}$ is an open cover of Y, let $\mathfrak{W}$ be an open cover such that for each $W \in \mathfrak{W}$ there is $U \in \mathfrak{U}$ such that $W^* \subset U$ and any map $S^k \to f^{-1}(W^*)$ extends to a map $B^{k+1} \to f^{-1}U$.

Let $\mathcal{V}$ be an open cover refining $\mathcal{W}$ such that both $E(\mathcal{V}, \mathcal{W}; k)$ and $H(\mathcal{V}, \mathcal{W}; k-1)$ hold. Then both $E(\mathcal{V}, \mathcal{U}; k+1)$ and $H(\mathcal{V}, \mathcal{U}; k)$ hold. In fact if K, L, g, h are as in the first assertion, there is an extension $g'': L\cup K^k \to X$ such that for every k-simplex β of $\mathrm{cl}(K-L)$ there is $W(\beta)\in\mathcal{W}$ with $fg''(\beta)\cup h(\beta)\subset W(\beta)$. Let α be a $(k+1)$-simplex of $K-L$, and let $W\in\mathcal{W}$ be such that $h(\alpha)\subset W$. If $\beta<\alpha$, then $W\cap W(\beta)\neq\emptyset$. It follows that $fg''(\partial\alpha)\subset W^*$; hence there is $U\in\mathcal{U}$ such that $W^*\subset U$, and there is an extension $\alpha\to f^{-1}U$ of $g''|\partial\alpha: \partial\alpha\to f^{-1}(W^*)$. Combining such extensions gives the desired map $g': K\to X$; thus $E(\mathcal{V}, \mathcal{U}; k+1)$ holds. On the other hand if K, L, g, g', g'' are as in the second assertion, let $G': (L\cup K^{k-1})\times I\to X$ be a homotopy as in $H(\mathcal{V}, \mathcal{W}; k-1)$, and extend G' to $G'': K\times\{0, 1\}\cup(L\cup K^{k-1})\times I\to X$ by $G''(p, 0)=g'(p)$ and $G''(p, 1)=g''(p)$ for all points $p\in K$. For any k-simplex α of K there is $W\in\mathcal{W}$ such that $G''(\alpha\times\{0, 1\})\subset f^{-1}W$; thus $G''(\partial(\alpha\times I))\subset f^{-1}(W^*)$. Then G'' extends to a homotopy $G: K\times I\to X$ which maps $K\times I$ into $\mathcal{U}$, which proves that $H(\mathcal{V}, \mathcal{U}; k)$ holds.

LEMMA 2. *Let Y be paracompact, and let $f: X\to Y$ be a local connection in dimension n. Then for any open cover $\mathcal{U}$ of Y there is an open cover $\mathcal{V}$ refining $\mathcal{U}$ such that $H(\mathcal{V}, \mathcal{U}; f; n)$ holds.*

PROOF. If $n=-1$, there is nothing to prove. Assume that the lemma is true for $n<k$, and let $f: X\to Y$ be a local connection in dimension k. Then f is a strong local connection in dimension $k-1$. If $\mathcal{U}$ is an open cover of Y, let $\mathcal{W}$ be an open cover such that for each $W\in\mathcal{W}$ there is $U\in\mathcal{U}$ such that $W^*\subset U$ and for any map $h: S^k\to f^{-1}(W^*)$ the map fh extends to a map $B^{k+1}\to U$. By Lemma 1 there is an open cover $\mathcal{V}$ of Y refining $\mathcal{W}$ such that $H(\mathcal{V}, \mathcal{W}; k-1)$ holds. To see that $H(\mathcal{V}, \mathcal{U}; f; k)$ holds consider K, L, g, g', and g'' as in the assertion, and let $G: (L\cup K^{k-1})\times I\to X$ be a homotopy as in $H(\mathcal{V}, \mathcal{W}; k-1)$. Extend G to a map $G': K\times\{0, 1\}\cup(L\cup K^{k-1})\times I\to X$ by $G'(p, 0)=g'(p)$ and $G'(p, 1)=g''(p)$, and observe that for every k-simplex α of $K-L$ there is $W\in\mathcal{W}$ with $fG'(\partial(\alpha\times I))\subset W^*$. It follows that fG' extends to a homotopy H which maps $K\times I$ into $\mathcal{U}$.

2. **Proof of Theorem 1.** Let Y have metric d. Using Lemmas 1 and 2 choose a sequence $\{\mathcal{V}_r | 0\leq r<\infty\}$ of open covers of Y such that mesh $\mathcal{V}_r<\epsilon/4(r+1)$, $\mathcal{V}^*_{r+1}$ refines $\mathcal{V}_r$, $E(\mathcal{V}_{r+1}, \mathcal{V}_r; n)$ holds, and $H(\mathcal{V}^*_{r+1}, \mathcal{V}_r; f; n)$ holds. (For a given ϵ such a sequence provides the extension and the homotopy in all cases.)

If K, L, g, and h are as in Theorem 1, choose a sequence $\{K_r | 1\leq r<\infty\}$ of subdivisions of K such that K_{r+1} is a subdivision of K_r and h maps K_r into $\mathcal{V}_{r+1}$. Using the fact that $E(\mathcal{V}_{r+1}, \mathcal{V}_r; n)$ holds choose extensions $g_r: K_r \to X$ of g $(1 \leqq r < \infty)$ such that for every α in K_r there is $V \in \mathcal{V}_r$ with $fg_r(\alpha) \cup h(\alpha) \subset V$. Since mesh $\mathcal{V}_r < \epsilon/4(r+1)$, $d(fg_r(p), h(p)) < \epsilon/4(r+1)$ for all points $p \in K$.

Set $g' = g_1$, and construct H by "filling in" between g_r and g_{r+1} as follows. Let α be a simplex of K_r, and let $fg_r(\alpha) \cup h(\alpha) \subset V$ for some $V \in \mathcal{V}_r$. Consider α as a subcomplex of K_{r+1}, and observe that for every simplex β of α, there is $V' \in \mathcal{V}_r$ such that $fg_{r+1}(\beta) \cup h(\beta) \subset V'$; hence $fg_{r+1}(\alpha) \cup fg_r(\alpha) \subset V^*$. Since $H(\mathcal{V}_r^*, \mathcal{V}_{r-1}; f; n)$ holds, there is a homotopy $H_r: fg_{r+1} \simeq fg_r$ relative to L, which may be considered as a map $H_r: K \times [1/(r+1), 1/r] \to Y$, such that the diameter of $H_r(\alpha \times [1/(r+1), 1/r])$ is $< \epsilon/4r$ for every simplex α of K_r. This implies that $d(H_r(p, t), h(p)) < \epsilon/2r$ for all $(p, t) \in K \times [1/(r+1), 1/r]$. Define $H: K \times I \to Y$ by $H(p, t) = H_r(p, t)$ for $1/(r+1) \leqq t \leqq 1/r$ and by $H(p, 0) = h(p)$. It is easy to check that H is a map and is in fact an ϵ-homotopy relative to L. This completes the proof.

3. **Proof of Theorem 2.** Since f is a closed map and $f^{-1}(y)$ is $(n-1)$-connected for every $y \in Y$, to show that f is a local connection in dimension n it suffices to show that for every open neighborhood U of $f^{-1}(y)$ there is an open neighborhood $V \subset U$ of $f^{-1}(y)$ such that for $0 \leqq k \leqq n$ any map $S^k \to V$ is homotopic in U to a map $S^k \to f^{-1}(y)$. Let A be the inverse set under f of a point of Y. Since for any closed LC^{n-1} subset A of an LC^n space X, the inclusion map $i: A \subset X$ is a local connection in dimension n, and since X is paracompact, Lemma 1 applies to $i: A \subset X$ and $1: X \subset X$. If U is an open neighborhood of A, let $\mathcal{U}$ be the cover of X consisting of U and $X - A$, let $\mathcal{W}$ be an open cover of X refining $\mathcal{U}$ such that $H(\mathcal{W}, \mathcal{U}; n)$ holds for $1: X \subset X$, and let $\mathcal{V}$ be an open cover refining $\mathcal{W}$ such that $E(\mathcal{V}, \mathcal{W}; n)$ holds for $i: A \subset X$.

Set $V = \bigcup \{ V' \in \mathcal{V} \mid V' \cap A \neq \varnothing \}$. For $0 \leqq k \leqq n$ triangulate S^k in some way as a complex K, and observe that for any map $h: K \to V$ there is a subdivision K' of K such that h maps K' into $\{ V' \in \mathcal{V} \mid V' \cap A \neq \varnothing \}$. It follows that there is a map $g: K' \to A$ such that for each α in K' there is $W \in \mathcal{W}$ with $g(\alpha) \cup h(\alpha) \subset W$. Since $H(\mathcal{W}, \mathcal{U}; n)$ holds for $1: X \subset X$, there is a homotopy $H: h \simeq g: K' \times I \to X$ such that H maps $K' \times I$ into $\mathcal{U}$. Since $g(K') \subset A$, $H(K' \times I) \subset U$. This completes the proof.

4. **Proof of Theorem 3.** We recall that M is an n-manifold, if it is a separable metric space each point of which has an open neighborhood homeomorphic to R^n and that a subset A of M is cellular, if $A = \bigcap_{j=1}^{\infty} Q_j^n$, where Q_j^n is a closed n-cell $(1 \leqq j < \infty)$ and int $Q_j^n \supset Q_{j+1}^n$.

If G is an upper semicontinuous decomposition of M into cellular subsets, then it is well known that M/G is a separable metric space and that the projection $P: M \rightarrow M/G$ is a closed map. It follows directly from the definition of cellularity that P is a strong local connection in dimension k for all k, and therefore that M/G is LC^∞. In order to prove that for every point $x \in M/G$ and every neighborhood N of x there are (connected) open neighborhoods V, U of x such that $V \subset U \subset N$ and for all k the image of $\pi_k(V-x)$ in $\pi_k(U-x)$ (under the homomorphism induced by the inclusion $V-x \subset U-x$) is isomorphic to $\pi_k(S^{n-1})$ we could duplicate the arguments of [1] using the fact that $P | P^{-1}(W)$ is a local connection in all dimensions for every open set W of M/G wherever Smale's theorem is used. We shall omit these details.

References

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