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On nonacyclicity of the quotient space of $\mathbb{R}^3$ by the solenoid

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Abstract

It is well-known that the quotient space of the 3-dimensional Euclidean space $\mathbb{R}^3$ by the dyadic solenoid is not simply connected. We prove that the singular homology of this quotient space is uncountable.

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1. Introduction

Bing [1] was the first to observe that the quotient space $\mathbb{R}^3/\Sigma_2$ of the 3-dimensional Euclidean space $\mathbb{R}^3$ by the dyadic solenoid Σ_2 has a nontrivial fundamental group (a complete proof of this result was first published in [8,9]). However, not much is known about its properties. Therefore it is of interest to understand the nature of this group.

The quotient space $\mathbb{R}^3/\Sigma_2$ is homotopy equivalent to the dyadic projective telescope $\mathcal{P}_2\mathcal{T}$. Bogley and Sieradski have shown that the fundamental group $\pi_1(\mathcal{P}_2\mathcal{T})$ is non-Abelian [2,11]. The purpose of the present paper is to show that the abelianization of the fundamental group $\pi_1(\mathbb{R}^3/\Sigma_\mathcal{P})$ of the quotient space $\mathbb{R}^3$ by any solenoid $\Sigma_\mathcal{P}$ is an *uncountable* group.

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Theorem 1.1. *The quotient space $\mathbb{R}^3/\Sigma_{\mathcal{P}}$ of $\mathbb{R}^3$ by any solenoid $\Sigma_{\mathcal{P}}$ is homotopy equivalent to the projective telescope $\mathcal{PT}$ and the singular homology group $H_1(\mathbb{R}^3/\Sigma_{\mathcal{P}};\mathbb{Z})$ is uncountable.*

2. Preliminaries

Let S^1 be the oriented unit circle in the complex plane $\mathbb{C}$. Consider the following inverse sequence $\mathcal{P}$:

$$P_0 \xleftarrow{f_0} P_1 \xleftarrow{f_1} P_2 \xleftarrow{f_1} \cdots$$

where P_0 is a point, P_k is the circle S^1 and $f_k : S^1 \to S^1$ is the standard continuous mapping of degree n_k, $n_k > 1$, for every $k > 0$. The inverse limit $\varprojlim \mathcal{P}$ is called the *solenoid* $\Sigma_{\mathcal{P}}$. The space $\Sigma_{\mathcal{P}}$ is one-dimensional, compact and metric. It has a standard embedding into $\mathbb{R}^3$ (see, e.g., [5, pp. 230–231]). If $n_k = 2$ for all k, then $\Sigma_{\mathcal{P}}$ is called the *dyadic solenoid* and denoted by Σ_2.

Let $C(f_0, f_1, f_2, \ldots)$ be the *infinite mapping cylinder* (see, e.g., [6,7,10]) and let $\widetilde{\mathcal{P}}$ be its natural compactification by the solenoid $\Sigma_{\mathcal{P}}$. The projective telescope $\mathcal{PT}$ is the one-point compactification of $C(f_0, f_1, f_2, \ldots)$ by some point $\{pt\}$. We consider $\{pt\}$ as the base point of $\mathcal{PT}$ and the circles P_k for $k = \{1, 2, 3, \ldots\}$ as the natural subspaces of $\mathcal{PT}$.

Hereafter, by homology we shall mean the singular homology with integer coefficients. Since the one-dimensional homology group of a path-connected space is the abelianization of the fundamental group, our results strengthen Bing's theorem mentioned above [1,8,9].

To prove Theorem 1.1 we shall need the following results:

Theorem 2.1 (Borsuk [3,9]). *Let W be a strong deformation retract of $\widehat{W}$ and let X be any continuum in W. Then W/X is a strong deformation retract of $\widehat{W}/X$. Thus in particular, W/X and $\widehat{W}/X$ have the same homotopy type.*

Proposition 2.2. *The compactum $\mathcal{PT}$ is an absolute retract.*

Proof. The proposition is a direct consequence of well-known results (see, e.g., [7, p. 104]). □

Consider the following closed subset of S^1:

$$A = \left\{ e^{2\pi i t} \in S^1 \mid t = \frac{1}{k},\ k \in \mathbb{N} \right\}.$$

The quotient space S^1/A is homeomorphic to the *Hawaiian earring* $\mathcal{H}$, i.e., to the compact bouquet of a countable number of circles $\{S^1_k\}_{k\in\mathbb{N}}$.

Let $p : S^1 \to \mathcal{H}$ be the canonical projection, $\mathbb{Z}$ the infinite cyclic group and $\mathbb{Z}_n$ the finite cyclic subgroup of order n of S^1:

$$\mathbb{Z}_n = \left\{ e^{2\pi i t} \in S^1 \mid t = \frac{k}{n},\ k = 1, 2, \ldots, n \right\}.$$

3. Proof of Theorem 1.1

Since the space $\widetilde{\mathcal{P}}$ is a 2-dimensional compactum, it can be considered as a closed subspace of $\mathbb{R}^5$. Since $\mathbb{R}^5$ and (by Proposition 2.2) $\widetilde{\mathcal{P}}$ is an absolute retract, $\widetilde{\mathcal{P}}$ is a strong deformation retract of $\mathbb{R}^5$. The compactum $\Sigma_{\mathcal{P}}$ is a subset of $\widetilde{\mathcal{P}}$, therefore by Theorem 2.1 the quotient space $\mathbb{R}^5/\Sigma_{\mathcal{P}}$ is homotopy equivalent to the quotient space $\widetilde{\mathcal{P}}/\Sigma_{\mathcal{P}}$, which is obviously homeomorphic to the projective telescope $\mathcal{PT}$.

Since the homotopy type of $\mathbb{R}^5/\Sigma_{\mathcal{P}}$ does not depend on the way in which $\Sigma_{\mathcal{P}}$ is embedded into $\mathbb{R}^5$ (see Theorem 1 in [9]), we can assume that $\Sigma_{\mathcal{P}}$ is embedded into $\mathbb{R}^5$ as the composition of the standard embeddings $\Sigma_{\mathcal{P}} \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R}^2$, where 0 is the origin of $\mathbb{R}^2$. By Theorem 2.1, $\mathbb{R}^3/\Sigma_{\mathcal{P}}$ is homotopy equivalent to $\mathbb{R}^5/\Sigma_{\mathcal{P}}$ and therefore to the projective telescope $\mathcal{PT}$. The first part of Theorem 1.1 is thus proved.

Suppose now that to the contrary, $H_1(\mathcal{PT})$ were a *countable* group. Consider $\mathcal{PT}$ as the union: $\mathcal{PT} = C(f_0) \cup C(f_1, f_2, f_3, \ldots)^*$, where $C(f_0)$ is the cylinder of the constant mapping $f_0 : S^1 \to S^1$ and therefore is a contractible space, and $C(f_1, f_2, f_3, \ldots)^*$ is the one-point compactification of the infinite mapping cylinder $C(f_1, f_2, f_3, \ldots)$. The intersection of these two subspaces of $\mathcal{PT}$ is the circle S^1. Thus it follows by the Mayer–Vietoris exact sequence:

$$\to H_1\big(S^1\big) \to H_1\big(C(f_0)\big) \oplus H_1\big(C(f_1, f_2, f_3, \ldots)^*\big) \to H_1(\mathcal{PT}) \to \cdots$$

that the group

$$H_1\big(C(f_1, f_2, f_3, \ldots)^*\big) \quad \text{is countable.} \tag{3.1}$$

Consider now $C(f_{n+1}, f_{n+2}, f_{n+3}, \ldots)^*$ as a subspace of $C(f_1, f_2, f_3, \ldots)^*$. Let X_n and $p_n : C(f_1, f_2, f_3, \ldots)^* \to X_n$ be the corresponding quotient space and the quotient mapping. For every sequence of units and zeros $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, let $g_\alpha : \mathcal{H} \to \mathcal{H}$ be the mapping such that

$$g_\alpha|_{S_k^1} = \begin{cases} \text{the identity mapping onto its image,} & \text{if } \alpha_k = 1, \\ \text{the constant mapping into the base point,} & \text{if } \alpha_k = 0. \end{cases}$$

Let g be a mapping of $\mathcal{H}$ to $C(f_1, f_2, f_3, \ldots)^*$ which maps the base point of $\mathcal{H}$ to the base point $\{pt\}$ of $C(f_1, f_2, f_3, \ldots)^*$ and such that the restriction $g|_{S_k}$ only wraps once around the circle P_k in the positive direction.

The set $\{g_\alpha\}$ is uncountable. However, the group $H_1(C(f_1, f_2, f_3, \ldots)^*)$ is countable (3.1). Therefore there exist two sequences α and β such that $\alpha \neq \beta$ and such that for the mappings $S^1 \xrightarrow{p} \mathcal{H} \xrightarrow{g_\alpha} \mathcal{H} \xrightarrow{g} C(f_1, f_2, f_3, \ldots)^*$ and $S^1 \xrightarrow{p} \mathcal{H} \xrightarrow{g_\beta} \mathcal{H} \xrightarrow{g} C(f_1, f_2, f_3, \ldots)^*$ we obtain the same homomorphism of the corresponding homology groups:

$$(gg_\alpha p)_1 = (gg_\beta p)_1 : H_1\big(S^1\big) \to H_1\big(C(f_1, f_2, f_3, \ldots)^*\big). \tag{3.2}$$

On the other hand, let m be the minimal number such that $\alpha_m \neq \beta_m$. To the projection $p_m : C(f_1, f_2, f_3, \ldots)^* \to X_m$ there correspond two homomorphisms of homology groups: $H_1(S^1) \xrightarrow{(p_m g g_\alpha p)_1} H_1(X_m)$ and $H_1(S^1) \xrightarrow{(p_m g g_\beta p)_1} H_1(X_m)$. Since $\alpha_k = \beta_k$ for $k < m$ and $\alpha_m \neq \beta_m$, by construction we have $(p_m g g_\alpha p)_1(1) \neq (p_m g g_\beta p)_1(1)$, contradicting (3.2).

Question 3.1. Let X be the Case–Chamberlin continuum [4]. Is then the homology of quotient space $H_1(\mathbb{R}^3/X)$ nontrivial?

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