Communicated 11 October 1950 by F. Carlson and J. Malmquist

Some theorems on absolute neighborhood retracts

By Olof Hanner

1. In this paper we shall study ANR's (absolute neighborhood retracts). The general problem will be as follows. Suppose we have proved that all ANR's have a certain property. Then we may ask, if this property is characteristic for ANR's, or in other words if it is true that a separable metric space having this property necessarily is an ANR. Thus we shall study necessary conditions for a space to be an ANR, and we shall find that some of these conditions are also sufficient.

Using Kuratowski's modification ([7] p. 270) of Borsuk's original definition ([1] p. 222), we mean by an ANR a separable metric space X such that, whenever X is imbedded as a closed subset of another separable metric space Z, it is a retract of some neighborhood in Z.

First, we take up the study of local properties of ANR's. It is known that an ANR is locally contractible (cf. [4] p. 273) and Borsuk proved that local contractability is sufficient for a finite dimensional compact space to be an ANR ([1] p. 240). In a recent paper, however, he has given an example of a locally contractible infinite dimensional space, which is not an ANR [3]. So the question then arises, if the property of a space to be an ANR is a local property. That the answer is affirmative is shown by theorem 3.3. In the case of a compact space this has already been proved by Yajima [10].

Thereafter we prove some theorems on homotopy of mappings into an ANR. Briefly the result can be stated by saying that two mappings of the same space into an ANR which are "near" enough to each other, are homotopic, and that if the homotopy is already given on a closed subset and is "small" enough, then this homotopy is extendable. For a compact ANR we can give an exact meaning to the words near and small in terms of some metric. But the uniformity structure implied by a metric does not seem to be a suitable tool for handling non-compact ANR's. Instead of a metric we therefore use open coverings of the space.

Borsuk proved [2] that any compact ANR X is dominated by a finite polyhedron P. This means that there exists two mappings $\varphi: X \to P$ and $\psi: P \to X$ such that $\psi\varphi: X \to X$ is homotopic to the identity mapping $i: X \to X$. We now prove that the polyhedron P and the mappings φ and ψ can be chosen so that this homotopy between $\psi\varphi$ and i is arbitrarily small, and we show that in this way we get a sufficient condition. This result is generalized in a natural way to non-compact spaces by using infinite locally finite polyhedra. Since these polyhedra are ANR's (see corollary 3.5) we thus see that any ANR is dominated by a locally compact ANR.

Finally we study a theorem by J. H. C. WHITEHEAD. By a new proof we are able to generalize it slightly.

2. Let us develop in this section some notations and well-known results, which we need in the sequel.

All spaces in this paper will be **separable metric** (or rather separable metrizable, since we often consider different metrics for the same space).

By a pair (Y, B) we mean a space Y and a closed subset B of Y. If (Y, B) is a pair and $F: Y \to X$ and $f: B \to X$ are two mappings into a space X such that $F(y) = f(y)$ for $y \in B$, we call F an extension of f to Y and f the restriction of F to B, denoted $f = F|B$. If F is only defined on some neighborhood of B in Y, F is called a neighborhood extension of f in Y.

Suppose X is a subset of Z. Then we distinguish between two mappings $f: B \to X$ and $g: B \to Z$ for which $f(b) = g(b)$ for all $b \in B$. This distinction is of importance, when we speak about extensions. That f is extendable to $Y \supset B$ implies that g is extendable, but the converse is not in general true. An extension of g will also be called an extension of f relative to Z, and an extension of f will be called an extension of g relative to X.

If $f(y, t): Y \times I \to X$ denotes a homotopy, I being the interval $0 \leqq t \leqq 1$, we shall also use the notation $f_t: Y \to X$, where f_t for each t is the mapping determined by $f_t(y) = f(y, t)$. When the notation $f_t(y)$ is used this will implicitely mean that the function is continuous in both variables y and t.

By an open covering $\alpha = \{U_\lambda\}$ of a space X we mean a family of open subsets U_λ of X, the union of the U_λ's being X. An open covering $\beta = \{V_\mu\}$ is said to be a refinement of $\alpha = \{U_\lambda\}$, if for each μ there is a λ such that $V_\mu \subset U_\lambda$. A covering $\alpha = \{U_\lambda\}$ is called star-finite, if for each λ we have $U_\lambda \cap U_\mu \neq \emptyset$ (= the void set) for only a finite number of μ. By a countable covering $\alpha = \{U_\lambda\}$ we mean a covering for which the index set $\{\lambda\}$ is countable. A countable covering can be written $\alpha = \{U_n\}$, $n = 1, 2, \ldots$ with the set of natural numbers as the index set. S. KAPLAN ([6] p. 249) has proved that every open covering of a separable metric space X has a countable star-finite refinement. A slightly different proof of this theorem will be derived from the method in the proof of theorem 3.3 (see remark 3.4).

By a locally finite polyhedron we mean a simplicial polyhedron with a countable number of simplices each meeting only a finite number of simplices. The polyhedron is topologized in the natural way by taking as open sets all sets that intersect any simplex in an open subset of that simplex. This topology makes a locally finite polyhedron into a locally compact separable metric space. We shall later see that it is an ANR.

A locally finite polyhedron can be topologically imbedded in a Hilbert space as follows. Let $\{p_n\}$ be the vertices of the polyhedron and $\{e_n\}$ the unite points of the countable number of coordinate axises of the Hilbert space. Define the topological mapping φ by setting $\varphi(p_n) = e_n$ and extending the mapping linearly for every simplex of the polyhedron.

For every covering of a space the nerve of the covering is the abstract simplicial complex whose vertices are the sets of the covering and in which $\{U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_r}\}$ is a simplex if and only if $\bigcap_{i=1}^{r} U_{\lambda_i}$ is non-void. The polyhedron corresponding to this complex is called the geometrical realization of the nerve of the covering. The geometrical realization is locally finite if and only if the covering is countable and star-finite.

For countable, star-finite coverings of a space X we consider the barycentric mapping of X into the geometrical realization of the nerve of the covering. If the geometrical realization is imbedded in the Hilbert space in the way described above, this mapping is defined by letting the image-point of a point $x \in X$ be the point $\{a_n(x)\}$, where

$$a_n(x) = \frac{d(x, X - U_n)}{\sum_{i=1}^{\infty} d(x, X - U_i)}.$$

It has been proved that a space X is an ANR if and only if for any pair (Y, B) and any mapping $f: B \to X$ there exists a neighborhood extension of f (cf. Fox [4]). This result will be used in this paper without any further reference.

Let X be a closed subset of an ANR Z. Then X is an ANR if and only if X is a neighborhood retract of Z. To be able to apply this when determining if a space X is an ANR, we want to imbed X as a closed set in an ANR Z. This is always possible. In fact, we can choose Z to be an AR (absolute retract). If X is compact, we simply imbed X in the Hilbert cube I_ω. For I_ω is an AR. For a non-compact space this method fails to work, because X will not be closed in I_ω. Wojdysławski [9] has proved, however, that any space X can be imbedded as a closed set in a space T which is a convex subset of a Banach space and which is therefore an AR.

Some use has been made within the theory of compact ANR's of the fact that the Hilbert cube I_ω is convex. To be able to prove the corresponding theorems in the non-compact case we shall use the space T just mentioned.

3. Lemma 3.1. *Any open subset of an ANR is an ANR.*

Proof. Let O be an open subset of the ANR X. Suppose there is given a pair (Y, B) and a mapping $f: B \to O$. We shall show that f has a neighborhood extension.

Since X is an ANR, f has a neighborhood extension relative to X, say $g: U \to X$, where U is a neighborhood of B. The set O is open so that $V = g^{-1}(O)$ is open in U, and because $B \subset V$, V is a neighborhood of B in Y. Define $F: V \to O$ by

$$F(y) = g(y) \qquad \text{for } y \in V.$$

Then F is an extension of f to the neighborhood V of B in Y.

This proves lemma 3.1.

Lemma 3.1 suggests the following concept.

Definition. A space X is called a *local ANR* if every point $x \in X$ has a neighborhood which is an ANR.

If a point x has a neighborhood which is an ANR, it follows from lemma 3.1 that given any neighborhood U of x, there is an open neighborhood of x contained in U which is an ANR. This justifies the name local ANR.

It is clear that an ANR is a local ANR. The converse is also true.

Theorem 3.2. *The two concepts ANR and local ANR are equivalent.*

Theorem 3.3. *A space which is the union of open ANR's is an ANR.*

Proofs. To prove theorem 3.2 we have to show that a local ANR is an ANR. Since every point of a local ANR has an open neighborhood which is an ANR, theorem 3.2 follows from theorem 3.3.

To prove theorem 3.3 let us consider two special cases and from them deduce the general case.

a) X is the union of *two* open ANR's: $X = O_1 \cup O_2$. Let $f: B \to X$ be a given mapping of a closed subset B of a space Y. We have to show that f has a neighborhood extension.

The two sets

$$F_1 = B - f^{-1}(O_2), \quad F_2 = B - f^{-1}(O_1)$$

are disjoint sets, closed in B and therefore also closed in Y. By the normality of Y take two disjoint open sets Y_1 and Y_2 in Y such that

$$Y_1 \supset F_1, \quad Y_2 \supset F_2.$$

Then $Y_0 = Y - (Y_1 \cup Y_2)$ is a closed set in Y. Set $B_i = Y_i \cap B$, $i = 0, 1, 2$. We have

$$(1) \qquad f(B_0) \subset O_1 \cap O_2,$$
$$f(B_1) \subset O_1, \quad f(B_2) \subset O_2.$$

B_0 is a closed subset of Y_0, and $O_1 \cap O_2$ is an ANR. Therefore (1) shows that there exists an extension of $f \mid B_0$ relative to $O_1 \cap O_2$ to an open neighborhood U_0 of B_0 in Y_0. This extension defined on U_0 and the original mapping f defined on B agree on $B_0 = U_0 \cap B$, so they together define a mapping $g: U_0 \cup B \to X$. Since

$$U_0 = (U_0 \cup B) \cap Y_0,$$

U_0 is closed in $U_0 \cup B$. B, being closed in Y, is also closed in $U_0 \cup B$. Therefore g is continuous.

We have

$$(2) \qquad g(U_0 \cup B_1) \subset O_1, \quad g(U_0 \cup B_2) \subset O_2,$$

$$(3) \qquad Y_0 - U_0 \text{ is closed in } Y.$$

The set $U_0 \cup B_1$ is closed in $U_0 \cup Y_1$. For

$$(U_0 \cup Y_1) - (U_0 \cup B_1) = Y_1 - B_1 = Y_1 - B$$

is open in Y. Since O_1 is an ANR, we therefore in view of (2) have an extension $g_1: U_1 \to O_1$ of $g \mid U_0 \cup B_1$ relative to O_1 to an open neighborhood U_1 of $U_0 \cup B_1$ in $U_0 \cup Y_1$. Because of (3) $U_0 \cup Y_1$, is open in $Y_0 \cup Y_1$, so that

$$(4) \qquad U_1 \text{ is open in } Y_0 \cup Y_1.$$

Similarly let $g_2 : U_2 \to O_2$ be an extension of $g \mid U_0 \cup B_2$ relative to O_2 to an open neighborhood U_2 of $U_0 \cup B_2$ in $U_0 \cup Y_2$. Because of (3) $U_0 \cup Y_2$ is open in $Y_0 \cup Y_2$, so that

$$(5) \qquad U_2 \text{ is open in } Y_0 \cup Y_2.$$

Set $U = U_1 \cup U_2$ and define $F : U \to X$ by

$$F(u) = g_1(u) \quad \text{for } u \in U_1,$$
$$F(u) = g_2(u) \quad \text{for } u \in U_2.$$

For $u \in U_0 = U_1 \cap U_2$ we have $g_1(u) = g_2(u) = g(u)$. Hence F is uniquely determined. We have

$$U_1 = U - Y_2, \qquad U_2 = U - Y_1,$$

so that U_1 and U_2 are closed in their union U. Thus F is continuous. F is an extension of f. Therefore we only have to prove that U is a neighborhood of B in Y.

In fact, U is open in Y. For

$$Y - U = [(Y_0 \cup Y_1) - U_1] \cup [(Y_0 \cup Y_2) - U_2]$$

is closed because of (4) and (5).

This proof is essentially the same as BORSUK's proof ([1] p. 226) in the case of the union of two closed ANR's whose intersection is an ANR.

b) X is the *countable* union of *disjoint* open ANR's: $X = \bigcup_{n=1}^{\infty} O_n$. Suppose X is imbedded as a closed subset of a space Z. Choose some metric for Z. Each O_n, being the complement of an open subset of X, is closed in X and so also in Z. Define a collection of disjoint open sets $\{G_n\}$ in Z such that G_n contains O_n. This can be done for instance by letting G_n be the set of all points of Z whose distance to O_n is less than to $X - O_n$. Since O_n is an ANR and is a closed subset of G_n, it is a retract of some open set $H_n \subset G_n$. Denote the retraction by $r_n : H_n \to O_n$. These retractions together define a retraction $r : \bigcup_{n=1}^{\infty} H_n \to X$ by

$$r(z) = r_n(z) \quad \text{for } z \in H_n.$$

Since $\bigcup_{n=1}^{\infty} H_n$ is an open subset of Z containing X, this proves the theorem in case b).

c) Now to prove the theorem in the general case, note that by LINDELÖF's covering theorem X is a countable union of open ANR's: $X = \bigcup_{n=1}^{\infty} O_n$. From the sequence O_n we construct some other sequences of open sets.

First, define U_n by

$$U_n = \bigcup_{i=1}^{n} O_i.$$

U_n is open in X, and by successive use of a) we see that U_n is an ANR. Furthermore

$$X = \bigcup_{n=1}^{\infty} U_n,$$

$$U_n \subset U_{n+1}.$$

Secondly, define $V_n \subset U_n$ to be set of all points of X having a distance $> \frac{1}{n}$ to $X - U_n$ (in some metric on X). V_n is open in X, $V_n \subset U_n$, so V_n is an ANR. We have

$$X = \bigcup_{n=1}^{\infty} V_n, \tag{6}$$

$$\overline{V}_n \subset V_{n+1}. \tag{7}$$

Finally, define W_n by

$$W_1 = V_1, \quad W_2 = V_2,$$

$$W_n = V_n - \overline{V}_{n-2} \quad \text{for } n \geqq 3. \tag{8}$$

Each W_n is open in X, $W_n \subset V_n$, so W_n is an ANR. From (7) and (8) we obtain

$$W_n \supset V_n - V_{n-1},$$

so that (6) implies

$$X = \bigcup_{n=1}^{\infty} W_n = \bigcup_{n=1}^{\infty} W_{2n-1} \cup \bigcup_{n=1}^{\infty} W_{2n}.$$

But $\bigcup_{n=1}^{\infty} W_{2n-1}$ and $\bigcup_{n=1}^{\infty} W_{2n}$ are unions of disjoint open ANR's. Thus they are themselves ANR's by b). X, now being a union of two open ANR's, is an ANR. This proves theorem 3.3 and so also theorem 3.2.

Remark 3.4. The method of this proof can be used to demonstrate S. KAPLAN's theorem that an arbitrary open covering of a separable metric space has a countable star-finite refinement. For take from a given covering in the same way as above a countable refinement $\{O_n\}$ and construct W_n. Then W_n is an open subset of $\bigcup_{i=1}^{n} O_i$. Hence the covering

$$\{W_n \cap O_i\}, \quad n = 1, 2, \ldots, i = 1, \ldots, n,$$

is a refinement of the given covering and is clearly countable and star-finite.

Corollary 3.5. *A locally finite polyhedron is an ANR.*

Proof. It is known that a finite polyhedron is an ANR. Hence a locally finite polyhedron is a local ANR and so an ANR by theorem 3.2.

4. Let $f, g : Y \to X$ be two mappings of a space Y into a space X. Let $d(x_1, x_2)$ be a metric for the space X.

Definition. If X is covered by $\alpha = \{U_\lambda\}$, f and g are called *α-near* if for each $y \in Y$ there is a U_λ such that $f(y) \in U_\lambda$, $g(y) \in U_\lambda$. If $\varepsilon > 0$ is given, f and g are called *ε-near* if $d(f(y), g(y)) < \varepsilon$ for each $y \in Y$.

Let $f_t : Y \to X$ be a homotopy.

Definition. If X is covered by $\alpha = \{U_\lambda\}$, f_t is called an *α-homotopy* if for each $y \in Y$ there is a U_λ such that $f_t(y) \in U_\lambda$ for $0 \leqq t \leqq 1$. If $\varepsilon > 0$ is given, f_t is called an *ε-homotopy* if for each y the set of the points $f_t(y)$, $0 \leqq t \leqq 1$, has a diameter less than ε.

Theorem 4.1. *Let X be an ANR and α a given open covering. Then there exists an open covering β, which is a refinement of α, such that given any pair (Y, B), any two β-near mappings $F_0, F_1 : Y \to X$, and any β-homotopy $f_t : B \to X$ between $f_0 = F_0 | B$ and $f_1 = F_1 | B$ then there exists an extension of this homotopy to all of Y, the extension being an α-homotopy between F_0 and F_1.*

Note the special case when B is void. Any two β-near mappings are α-homotopic.

That the converse of theorem 4.1 is true, is shown by theorem 4.2.

Proof. As in section 2 we consider X as a subset of the space T. X being an ANR, there exists a retraction $r : U \to X$ of an open set U, $X \subset U \subset T$.

From the given covering $\alpha = \{U_\lambda\}$ we construct β as follows. Obviously $\alpha' = \{r^{-1}(U_\lambda)\}$ is an open covering of U. Let $\beta' = \{V_\mu\}$ be a refinement of α' such that each V_μ is convex. Put

$$\beta = \{V_\mu \cap X\}.$$

Then β is a refinement of α. We shall prove that β has the property stated in the theorem.

Let (Y, B) be a pair, F_0, F_1 two β-near mappings, and f_t a β-homotopy between $f_0 = F_0 | B$ and $f_1 = F_1 | B$. Since T is convex, the two points $F_0(y)$ and $F_1(y)$ can be joined in T by the straight line segment (using vector notation in T)

$$G_t(y) = (1 - t) F_0(y) + t F_1(y),$$

described by t going from 0 to 1. As we shall show below $G_t(y) \in U$. Hence $r G_t(y) : Y \to X$ is defined and is a homotopy between $F_0(y)$ and $F_1(y)$. But it is in general not an extension of f_t. We therefore want to replace $r G_t(y)$ by $f_t(y)$ for $y \in B$. To save the continuity we proceed as follows.

Define

$$h(y, t) : Y \times \{0\} \cup Y \times \{1\} \cup B \times I \to X$$

by

$$h(y, 0) = F_0(y) \quad \text{for } y \in Y,$$

$$h(y, 1) = F_1(y) \quad \text{for } y \in Y,$$

$$h(y, t) = f_t(y) \quad \text{for } y \in B,\ t \in I.$$

Since X is an ANR and I is compact there exists an extension $H(y, t)$ of $h(y, t)$ to a set $Y \times \{0\} \cup Y \times \{1\} \cup V \times I$, where V is an open neighborhood of B. By taking V small enough the homotopy $H_t \mid V : V \to X$ will be a β-homotopy like f_t.

Let W be an open set in Y such that $B \subset W \subset \overline{W} \subset V$, and define a function $e(y) : Y \to I$ such that

$$e(y) = 0 \quad \text{for } y \in Y - W,$$
$$e(y) = 1 \quad \text{for } y \in B.$$

Set

$$G'_t(y) = (1 - e(y))\, G_t(y) + e(y)\, H_t(y) \qquad \text{for } y \in V,$$
$$G'_t(y) = G_t(y) \qquad \text{for } y \in Y - V.$$

Then $G'(y, t)$ is continuous. We have

$$G'_0(y) = F_0(y), \qquad G'_1(y) = F_1(y).$$

Let us show that

(1) $$G'_t(y) \in U$$

and that for each y there is a $U_\lambda \in \alpha$ such that

(2) $$r\, G'_t(y) \in U_\lambda, \qquad \text{for } t \in I.$$

From (2) it will follow that $r\, G'_t(y) : Y \to X$ is an α-homotopy. Being an extension of $f_t(y)$, $r\, G'_t(y)$ is therefore the sought-for homotopy between $F_0(y)$ and $F_1(y)$.

We have to prove (1) and (2). They will follow if we show that for each $y \in Y$ the curve

(3) $$G'_t(y) \in V_\mu$$

for some $V_\mu \in \beta'$. There are two cases.

a) $e(y) = 0$. Then $G'_t(y) = G_t(y)$, the straight line segment between $F_0(y)$ and $F_1(y)$. Since F_0 and F_1 are β-near, and since all V_μ's are convex, this line segment lies in some $V_\mu \in \beta'$.

b) $e(y) > 0$. Then $y \in W$. $H_t(y)$ is a β-homotopy. Hence

(4) $$H_t(y) \in V_\mu$$

for some $V_\mu \in \beta'$. In particular

$$H_0(y) = F_0(y) = G_0(y) \in V_\mu, \quad H_1(y) = F_1(y) = G_1(y) \in V_\mu.$$

Since V_μ is convex we therefore deduce

(5) $$G_t(y) \in V_\mu,$$

and (3) follows from (4) and (5).

This completes the proof of theorem 4.1.

That ANR's are locally contractible is an immediate corollary of theorem 4.1.

Theorem 4.2. *A necessary and sufficient condition for a space X to be an ANR is that there exists an open covering α of X with the following property. If (Y, B) is a pair, $F_0, F_1 : Y \to X$ are two α-near mappings, and $f_t : B \to X$ is an α-homotypy between $f_0 = F_0 | B$ and $f_1 = F_1 | B$, then there exists a homotopy $F_t : Y \to X$ between F_0 and F_1 which is an extension of f_t.*

Proof. The necessity follows from theorem 4.1. To prove the sufficiency let $x \in X$ be an arbitrary point and let U be an element of the covering α containing x. Define $F'_0 : U \to X$, $F'_1 : U \to X$, and $f'_t : x \to X$ by

$$F'_0(u) = x \quad \text{for } u \in U,$$

$$F'_1(u) = u \quad \text{for } u \in U,$$

$$f'_t(x) = x \quad \text{for } t \in I.$$

Then obviously F'_0 and F'_1 are α-near, and f'_t is an α-homotopy, so we have an extension $F'_t : U \to X$ of f'_t.

Since $F'(x \times I) = x \in U$ we can by the compactness of I take an open neighborhood V of x such that

$$F'(V \times I) \subset U.$$

We assert that V is an ANR.

For let (Y, B) be any pair and $f : B \to V$ any mapping. Define $F''_0 : Y \to X$, $F''_1 : Y \to X$, and $f''_t : B \to X$ by

$$F''_0(y) = F''_1(y) = x \quad \text{for } y \in Y,$$

$$f''_t(y) = F'_{2t}(f(y)) \quad \text{for } y \in B,\ 0 \leqq t \leqq \tfrac{1}{2}$$

$$f''_t(y) = F'_{2-2t}(f(y)) \quad \text{for } y \in B,\ \tfrac{1}{2} \leqq t \leqq 1.$$

We see that F''_0 and F''_1 are trivially α-near and that f''_t is an α-homotopy. Then there exists an extension $F''_t : Y \to X$ of f''_t. Denote by W the open set $F''^{-1}_{\frac{1}{2}}(V)$ and define $F : W \to V$ by

$$F(y) = F''_{\frac{1}{2}}(y) \quad \text{for } y \in W.$$

Then F is a neighborhood extension of f, showing that V is an ANR.

Thus every point $x \in X$ has a neighborhood V which is an ANR, so X is an ANR by theorem 3.2. This proves theorem 4.2.

5. Definition. *The homotopy extension theorem* is said to *hold for a space X*, if for any homotopy $f_t : B \to X$ between two mappings $f_0, f_1 : B \to X$, where B is a closed subset of a space Y, the fact that f_0 is extendable to a mapping $F_0 : Y \to X$ implies that f_t is extendable to a homotopy $F_t : Y \to X$ between F_0 and an extension F_1 of f_1.

In particular let $f_0, f_1 : B \to X$ be two homotopic mappings. Then if f_0 is extendable to Y, f_1 is also extendable to Y.

It is known that the homotopy extension theorem holds for any ANR (cf. [5] p. 86).

Theorem 5.1. *A space X is an ANR if and only if for each point $x \in X$ there exists a neighborhood V of x such that for any pair (Y, B) any mapping $f : B \to V$ has an extension $F : Y \to X$ relative to X.*

Proof. The necessity is contained in a proof by KURATOWSKI ([7] p. 275). An alternative proof is the following. (Cf. also [10] p. 59.)

Let x be a point in the ANR X. X is locally contractible. Take V a neighborhood of x which in X is contractible to x. Any mapping $f : B \to V$ is then homotopic to the constant mapping $g : B \to V$, which maps all B into x. Since g is extendable to Y, f is extendable to Y relative to X by the homotopy extension theorem for X.

To prove the sufficiency we may assume V to be open, otherwise replacing V by any open neighborhood of x contained in V. We assert that V is an ANR. For let (Y, B) be a pair. A mapping $f : B \to V$ has then an extension $F : Y \to X$ relative to X. $W = F^{-1}(V)$ is open in Y. Then $f' : W \to V$ defined by

$$f'(y) = F(y) \qquad \text{for } y \in W$$

is a neighborhood extension of f. Hence V is an ANR, so that X is a local ANR and therefore also an ANR.

Remark 5.2. We may also prove that if U is a given neighborhood of x, we can choose V in theorem 5.1 so that we can require $F(Y) \subset U$ (cf. YAJIMA [10]). For we may assume U open. Then U is an ANR by lemma 3.1, and we can apply theorem 5.1 on U instead of on X.

Theorem 5.3. *If the homotopy extension theorem holds for a locally contractible space X, then X is an ANR.*

Proof. Let X be such a space. Theorem 5.1 gives a necessary and sufficient condition for a space to be an ANR. When proving the necessity of that condition, we only used the facts that an ANR is locally contractible and that for an ANR the homotopy extension theorem holds. Thus our space X satisfies that condition. Since the condition is also sufficient, X is an ANR.

The previously mentioned example by BORSUK [3] shows that there are locally contractible spaces which are not ANR's. As an example of a space for which the homotopy extension theorem holds but which is not an ANR, we can take the set of rationals on the real line.

6. In section 3 we proved that a locally finite polyhedron is an ANR. We are now going to show that any ANR is dominated by a locally finite polyhedron (see theorem 6.1). Later we shall prove that the converse of theorem 6.1 is true (see theorem 7.2).

Definition. The space Z is said to *dominate* the space X if there exist two mappings $\varphi : X \to Z$ and $\psi : Z \to X$ such that $\psi\varphi \simeq i : X \to X$, where i denotes the identity mapping. If the homotopy is an α-homotopy, α being a covering of X, Z is said to *α-dominate* X. If the homotopy is an ε-homotopy, ε being a positive number, Z is said to *ε-dominate* X.

Theorem 6.1. *If X is an ANR, then for any open covering α there exists a locally finite polyhedron α-dominating X.*[1]

Proof. As in section 2 we consider X as a subset of the space T. X being an ANR, there exists a retraction $r: U \to X$ of an open set $U, X \subset U \subset T$.

Let $\alpha = \{U_\lambda\}$ be the given covering. Consider $\alpha' = \{r^{-1}(U_\lambda)\}$, which is an open covering of U. For each $u \in U$ we determine a number $\eta = \eta(u) > 0$ such that the convex η-neighborhood $S(u, \eta)$ of u in T, i.e. the set of all points of T with a distance to u less than η, satisfies

$$S(u, \eta) \subset r^{-1}(U_\lambda) \tag{1}$$

for some $U_\lambda \in \alpha$. Set

$$\beta = \left\{S\left(u, \frac{\eta}{2}\right) \cap X\right\}.$$

Then β is a refinement of α. Let $\gamma = \{V_n\}$, $n = 1, 2, \ldots$, be a countable star-finite refinement of β. That γ is a refinement of β means that for each V_n we can select a point u_n and the corresponding number η_n such that

$$V_n \subset S\left(u_n, \frac{\eta_n}{2}\right) \cap X.$$

Let now P be the geometrical realization of the nerve of γ. Denote the vertex corresponding to V_n by p_n. We are going to show that P α-dominates X.

We define $\varphi: X \to P$ to be the barycentric mapping and set $\psi = r g: P \to X$, where $g: P \to U$ is the mapping defined by setting $g(p_n) = u_n$ for every vertex and extending the mapping linearly on every simplex of P. We have to verify that $g(P) \subset U$.

For an arbitrary point $x \in X$ denote by $V_{n_1}, \ldots, V_{n_r}$ the finite number of elements of γ containing x. Then $\eta_{n_1}, \ldots, \eta_{n_r}$ are the corresponding η-numbers, and we may assume that the notation is chosen so that $(i = 1, \ldots, r)$

$$\eta_{n_1} \geqq \eta_{n_i}.$$

From

$$x \in V_{n_i} \subset S\left(u_{n_i}, \frac{\eta_{n_i}}{2}\right) \cap X$$

we obtain

$$d(x, u_{n_i}) < \frac{\eta_{n_i}}{2} \leqq \frac{\eta_{n_1}}{2},$$

so that

$$d(u_{n_1}, u_{n_i}) < \eta_{n_1}. \tag{2}$$

The point x is mapped by φ into the simplex of P spanned by $p_{n_1}, \ldots, p_{n_r}$, and this simplex is mapped by g onto the simplex in T spanned by the points u_{n_i}. From (2) we see that

[1] I have been informed that C. H. Dowker has independently proved this theorem and its converse.

(3) $$u_{n_i} \in S(u_{n_1}, \eta_{n_1}).$$

Hence

(4) $$g\varphi(x) \in S(u_{n_1}, \eta_{n_1}).$$

Now to prove that $g(P) \subset U$ let $\sigma = (p_{m_1}, \ldots, p_{m_q})$ be a simplex of P. Let x in the arguments above be a point in $\bigcap_{j=1}^{q} U_{m_j}$. Then the p_{m_j}'s are among the p_{n_i}'s, and (3) yields

$$g(p_{m_j}) \in S(u_{n_1}, \eta_{n_1}).$$

Hence

$$g(\sigma) \subset S(u_{n_1}, \eta_{n_1}) \subset U,$$

showing $g(P) \subset U$.

Thus $\psi = rg$ is defined. Let us show that $\psi\varphi : X \to X$ is α-homotopic to the identity mapping $i : X \to X$. An arbitrary point $x \in X$ and the corresponding point $g\varphi(x)$ can be joined in T by a straight line segment. This gives a homotopy in T between $g\varphi$ and i. For each x, (4) and the trivial fact

(5) $$x \in S(u_{n_1}, \eta_{n_1})$$

show that this homotopy is in U. Applying r to the homotopy we have a homotopy in X between $\psi\varphi$ and i. (1), (4), and (5) imply that this is an α-homotopy.

Corollary 6.2. *Any ANR is dominated by a locally compact ANR.*

Proof. For a locally finite polyhedron is locally compact.

Theorem 6.3. *If X is a compact ANR with a given metric, then for any number $\varepsilon > 0$ there exists a finite polyhedron ε-dominating X.*

Proof. Let α in the proof of theorem 6.1 be a covering by open sets with diameter less than ε. Proceed as in that proof, but choose γ to be a finite covering. This we can do, since X is compact. The geometrical realization of the nerve of γ is a finite polyhedron ε-dominating X.

7. The purpose of this section is to prove the converse of theorem 6.1 (proved by theorem 7.2).

Let X be a space. We consider deformations of X, i.e. homotopies $h_t : X \to X$ such that $h_0 = i : X \to X$, the identity mapping. The mapping h_1 is a mapping into X. Thus if X is an ANR, there exists for any space Z in which X is imbedded as a closed subset, a neighborhood extension of h_1 in Z. Conversely, however, suppose that we know of a space X that there exists a deformation $h_t : X \to X$ such that whenever X is a closed subset of a space Z, h_1 is always extendable to some neighborhood of X. Then X is not necessarily an ANR. For any contractible space satisfies this condition, and there are contractible spaces which are not ANR's. However, if h_t can be chosen arbitrarily small, it turns out that X must be an ANR.

Definition. Let X be a space. A sequence of deformations $h_t^n : X \to X$, $h_0^n = i : X \to X$, $n = 1, 2, \ldots$, is said to *converge to the identity mapping* i, if for any point $x_0 \in X$ and any neighborhood V of x_0 there is another neigborhood W of x_0 and an integer N such that $x \in W$ and $n \geqq N$ imply $h^n(x, t) \in V$ for all t.

Theorem 7.1. *Each of the following three conditions is a sufficient condition for a space X to be an ANR. Let Z be a space in which X is imbedded as a closed subset.*

(a) *For each covering α of X there exists an α-deformation $h_t : X \to X$ such that for any Z, h_1 is extendable to a neighborhood of X in Z.*

(b) *For some metric on X there exists for each $\varepsilon > 0$ an ε-deformation $h_t : X \to X$ such that for any Z, h_1 is extendable to a neighborhood of X in Z.*

(c) *There exists a sequence of deformations $h_t^n : X \to X$ converging to the identity mapping such that for any Z, h_1^n is extendable to a neighborhood of X in Z.*

Proof. It is clear that of these conditions (a) implies (b) and (b) implies (c). Hence we have to prove that (c) is a sufficient condition. Note that when X is compact (a), (b), and (c) are directly seen to be equivalent.

Let $h_t^n : X \to X$ be as in (c). If (Y, B) is any pair and $f : B \to X$ is any mapping, then $h_1^n f$ is extendable to a neighborhood of B in Y. For, as in section 2, imbed X in T. Let n be fixed. The mapping $h_1^n : X \to X$ has a neighborhood extension $H : U \to X$, U open in T, and $f : B \to X$ has an extension $F : Y \to T$ relative to T. The set $V = F^{-1}(U)$ is then a neighborhood of B in Y, and $g : V \to X$ defined by $g(y) = HF(y)$ for $y \in V$ is a neighborhood extension of $h_1^n f$ in Y.

We want to prove that (c) implies that X is an ANR. Then it is enough to show that X is a neighborhood retract of T. We notice that $h_1^1 : X \to X$ has an extension $H_0 : U \to X$, U being a neighborhood of X in T. Let us show that X is a retract of U.

For that purpose we define a mapping $H(u, t) : U \times [0, 1) \to X$, where $[0, 1)$ denotes the half-open interval $0 \leqq t < 1$. Set

$$s_n = 1 - \frac{1}{2^{n-1}}, \quad n = 1, 2, \ldots.$$

Starting with $H_{s_1} = H_0 : U \to X$ already defined as an extension of h_1^1, we successively define H on the sets $U \times [s_n, s_{n+1}]$ by an induction on n. This will be done in such a way that $H_{s_n} | X = h_1^n$.

Assume H_t defined for $s_1 \leqq t \leqq s_n$. Take the space $U \times I$ and consider the closed subset $C = U \times \{0\} \cup X \times I$. Since $H_{s_n} | X = h_1^n$, we can map C into X by $g : C \to X$ defined by

$$g(u, 0) = H_{s_n}(u) \qquad \text{for } u \in U,$$

$$g(u, t) = h_{1-t}^n(u) \qquad \text{for } u \in X, \ t \in I.$$

The mapping $h_1^{n+1} g : C \to X$ is then, as we have proved, extendable to some neighborhood of C in $U \times I$. Applying DOWKER's method (cf. [5] p. 86), we get an extension $G: U \times I \to X$ of $h_1^{n+1} g$ defined on all of $U \times I$. Now define $H(u, t)$ for $t \in [s_n, s_{n+1}]$ by

$$H(u, (1-s) s_n + s\, s_{n+1}) = h_{2s}^{n+1} H_{s_n}(u) \qquad \text{for } 0 \leqq s \leqq \tfrac{1}{2},$$

$$H(u, (1-s) s_n + s\, s_{n+1}) = G(u, 2s-1) \qquad \text{for } \tfrac{1}{2} \leqq s \leqq 1.$$

This extends the definition of H to $U \times [s_n, s_{n+1}]$. We easily verify that H is uniquely determined for $t = s_n$ and $t = \frac{1}{2}(s_n + s_{n+1})$. For $t = s_{n+1}$ and $x \in X$ we have

$$H(x, s_{n+1}) = G(x, 1) = h_1^{n+1} g(x, 1) = h_1^{n+1}(x),$$

which means that

$$H_{s_{n+1}} | X = h_1^{n+1}.$$

Therefore the induction works. In this way H will be defined for all of $U \times [0, 1)$. H is clearly continuous.

Our next step will be to extend the mapping $K_t = H_t | X : X \to X$ defined on $X \times [0, 1)$ to all of $X \times I$ by setting $K(x, 1) = x$. Then $K(x, t)$ is a continuous function. This is already proved for $0 \leqq t < 1$, and is proved for $t = 1$ as follows. Let x_0 be any point in X and V any neighborhood of x_0 in X. We want to find a neighborhood $W \times [T, 1]$ of $x_0 \times \{1\}$ such that

$$(1) \qquad K(W \times [T, 1]) \subset V.$$

For $t \in [s_n, s_{n+1}]$ K_t takes the values $(t = (1-s) s_n + s\, s_{n+1})$:

$$K_t(x) = h_{2s}^{n+1} h_1^n(x) \qquad \text{for } 0 \leqq s \leqq \tfrac{1}{2},$$

$$K_t(x) = h_1^{n+1} h_{2-2s}^n(x) \qquad \text{for } \tfrac{1}{2} \leqq s \leqq 1,$$

i.e. values of the form $h_{t'}^{n+1} h_{t''}^n(x)$. Since the sequence of mappings $h_{t'}^{n+1}$ converges to the identity mapping, we can find a neighborhood W_1 of x_0 and an integer N_1 such that

$$(2) \qquad h_{t'}^{n+1}(x) \in V \qquad \text{for } x \in W_1,\ n \geqq N_1.$$

Again, since $h_{t''}^n$ converges to the identity mapping, we can take a neighborhood W of x_0 and an integer $N \geqq N_1$ such that

$$(3) \qquad h_{t''}^n(x) \in W_1 \qquad \text{for } x \in W,\ n \geqq N.$$

Hence setting $T = s_N$ we have for $(x, t) \in W \times [T, 1]$ that

$$(4) \qquad K(x, t) \subset V.$$

For, if $T \leqq t < 1$, $t \in [s_n, s_{n+1}]$ for some $n \geqq N$, and (2) and (3) imply (4). Since $W \subset W_1 \subset V$ and $K(x, 1) = x$, (4) is also true when $t = 1$. This proves (1) and shows the continuity of $K(x, t)$ for $t = 1$.

The two functions $H : U \times [0, 1) \to X$ and $K : X \times I \to X$, which are both continuous, agree on $X \times [0, 1)$. However, they do not in general define a continuous function on $U \times [0, 1) \cup X \times \{1\}$, since for a sequence $u_n \in U - X$ for which $u_n \to x \in X$, and a sequence $t_n \to 1$, we do not necessarily have $H(u_n, t_n) \to K(x, 1) = x$. Therefore, when we finally define the retraction $r : U \to X$ by setting

(5)
$$r(u) = H(u, e(u)) \quad \text{for } u \in U - X,$$
$$r(u) = u \quad \text{for } u \in X,$$

where $0 \leqq e(u) < 1$ is a function tending to 1, when u approaches to X, we have to be careful when choosing $e(u)$, so that r is continuous.

Let $d(u_1, u_2)$ be a metric on U and take the metric

$$d_1((u_1, t_1), (u_2, t_2)) = d(u_1, u_2) + |t_1 - t_2|$$

for $U \times [0, 1)$. In $U \times [0, 1)$ consider the open neighborhood V_0 of $X \times [0, 1)$ defined as follows. A point (u, t) belongs to V_0 if and only if there is a point $(x, t') \in X \times [0, 1)$ such that

(6)
$$d_1((u, t), (x, t')) < 1 - t,$$

(7)
$$d(H(u, t), H(x, t')) < 1 - t.$$

That V_0 is open is clear from the fact that the same point (x, t') can be used for a neighborhood of (u, t).

Since for each $n = 1, 2, \ldots$ $[s_n, s_{n+1}]$ is compact, $X \times [s_n, s_{n+1}]$ is contained in a subset of V_0 of the form $U_n \times [s_n, s_{n+1}]$, U_n being an open subset of U. We may assume $U_n \supset \overline{U}_{n+1}$. Take a mapping $e_n : U \to I$ such that

$$e_n(u) = 0 \quad \text{for } u \in U - U_n,$$
$$e_n(u) = 1 \quad \text{for } u \in \overline{U}_{n+1}.$$

Set

$$e(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n(u).$$

This function $e(u)$ has the following properties.

(8)
$$e(u) : U \to I \quad \text{is continuous},$$

(9)
$$e(u) = 1 \quad \text{for } u \in X,$$

(10)
$$e(u) = 0 \quad \text{for } u \in U - U_1,$$

(11)
$$(u, e(u)) \in V_0 \quad \text{for } u \in U_1 - X.$$

Properties (8), (9), and (10) follow directly from the definition of $e(u)$. To prove (11) we observe that (6) implies

$$\bigcap_{n=1}^{\infty} U_n = X,$$

so that if $u \in U_1 - X$, we have $u \in U_i - U_{i+1}$ for some i. Then

$$e_n(u) = 1 \quad \text{for } n < i,$$
$$e_n(u) = 0 \quad \text{for } n > i,$$

so that

$$e(u) = \frac{1}{2} + \cdots + \frac{1}{2^{i-1}} + \frac{1}{2^i} e_i(u)$$
$$= 1 - \frac{1}{2^{i-1}} + \frac{1}{2^i} e_i(u).$$

Therefore

$$e(u) \in [s_i, s_{i+1}],$$

and

$$(u, e(u)) \in U_i \times [s_i, s_{i+1}] \subset V_0.$$

With this function $e(u)$ define $r : U \to X$ by (5). Obviously r is a retraction. It only remains to be proved that r is continuous. All we have to show is that if we have a sequence $u_n \in U_1 - X$, $n = 1, 2, \ldots$, then $u_n \to x \in X$ implies $r(u_n) \to r(x) = x$.

The point $(u_n, e(u_n))$ belongs to V_0, so there is a point $(x_n, t_n') \in X \times [0, 1)$ such that $(u_n, e(u_n))$ and (x_n, t_n') satisfy (6) and (7). From $u_n \to x$ we obtain $e(u_n) \to e(x) = 1$, so that

$$(12) \qquad (u_n, e(u_n)) \to (x, 1),$$

and (6) and (7) yield

$$(13) \qquad d_1((u_n, e(u_n)), (x_n, t_n')) \to 0,$$

$$(14) \qquad d(H(u_n, e(u_n)), H(x_n, t_n')) \to 0.$$

From (12) and (13)

$$(x_n, t_n') \to (x, 1).$$

Hence

$$(15) \qquad K(x_n, t_n') \to K(x, 1) = x.$$

Since, however, $K(x_n, t_n') = H(x_n, t_n')$, we obtain from (14) and (15)

$$r(u_n) = H(u_n, e(u_n)) \to x,$$

showing the continuity of r.

This completes the proof of theorem 7.1.

Theorem 7.2. *Each of the following three conditions is a sufficient condition for a space X to be an ANR.*

(a) *For each covering* α *of* X *there exists an ANR* α*-dominating* X.

(b) *For some metric on* X *there exists for each* $\varepsilon > 0$ *an ANR* ε*-dominating* X.

(c) *There exists a sequence of ANR's* Z_n *dominating* X *such that the corresponding sequence of homotopies* $\psi_n \varphi_n \simeq i : X \to X$ *is a sequence of deformations converging to the identity mapping.*

Proof. Let X be imbedded in Z as a closed subset, and let Z' be an ANR dominating X, $\varphi : X \to Z'$, $\psi : Z' \to X$, $\psi \varphi \simeq i : X \to X$. The theorem will be an immediate consequence of theorem 7.1, if we can show that $\psi \varphi$ has an extension to some neighborhood of X in Z.

But (Z, X) is a pair and $\varphi : X \to Z'$ is a mapping into an ANR, so there is an extension $\Phi : U \to Z'$, U being a neighborhood of X in Z. Then $\psi \Phi : U \to X$ is a neighborhood extension of $\psi \varphi$. This proves theorem 7.2.

In particular theorem 7.2 contains the converse of theorem 6.1. That the sufficient conditions given in theorem 7.1 and theorem 7.2 for a space X to be an ANR also are necessary, is trivial.

8. In this final section we shall use theorem 7.1 to give a new proof of a theorem by J. H. C. WHITEHEAD [8]. At the same time we shall be able to slightly generalize the theorem, in that we do not require all spaces to be compact.

Lemma 8.1. *Let* (X_1, A_1) *be a pair such that* X_1 *and* A_1 *are ANR's. Then if* α *is a covering of* X_1, *there exists an* α*-deformation* $k_t : X_1 \to X_1$ *satisfying*

(1) $$k(x, 0) = x \quad \text{for } x \in X_1,$$

(2) $$k(x, t) = x \quad \text{for } x \in A_1,\ t \in I,$$

(3) *there exists an open set* V, $A_1 \subset V \subset X_1$, *for which* $k_1(V) = A_1$.

Proof. Since A_1 is an ANR we can find a retraction $r : U \to A_1$, where U is a closed neighborhood of A_1 in X_1. In the space $X_1 \times I$ we consider the closed subset

$$D = X_1 \times \{0\} \cup U \times \{1\} \cup A_1 \times I$$

and the mapping $g : D \to X_1$ defined by

$$\begin{aligned} g(x, 0) &= x && \text{for } x \in X_1, \\ g(x, 1) &= r(x) && \text{for } x \in U, \\ g(x, t) &= x && \text{for } x \in A_1,\ t \in I. \end{aligned}$$

X_1 is an ANR. Hence there is a neighborhood extension of g to a function $G : E \to X_1$. E contains a set of the form $U' \times I$, where U' is an open set in X_1 containing A_1. We may choose U' so that $U' \subset U$ and so that $G_t \mid U'$ is an α-homotopy.

Now, let V be an open set in X_1 such that $A_1 \subset V \subset \overline{V} \subset U'$. Take a function $e : X_1 \to I$ such that

$$\begin{aligned} e(x) &= 0 && \text{for } x \in X_1 - U', \\ e(x) &= 1 && \text{for } x \in \overline{V}. \end{aligned}$$

Set

$$k(x, t) = G(x, t\,e(x)) \quad \text{for } x \in X_1,\ t \in I.$$

Then $k(x, t)$ is immediately seen to be an α-deformation satisfying (1) and (2). (3) follows from

$$k(x, 1) = G(x, 1) = g(x, 1) = r(x) \in A_1 \quad \text{for } x \in V.$$

This proves lemma 8.1.

Let X_1 and X_2 be two ANR's, and let there be given a mapping $\varphi : A_1 \to X_2$, where $A_1 \subset X_1$ is a compact ANR. Observe that we do not require X_1 and X_2 to be compact.

We will introduce a new space X which we shall prove to be an ANR. We may assume that X_1 and X_2 are disjoint open subsets of a space $Z = X_1 \cup X_2$. Identify in Z each point $a \in A_1$ with $\varphi(a) \in X_2$. The identification space thus obtained from Z is called X.

Theorem 8.2. *X is an ANR* (cf. [8]).

Proof. First we notice that X is a separable metric space. This is proved by elementary arguments, using the fact that A_1 is compact. We leave the details to the reader.

Denote the natural mapping of Z onto X by $\psi : Z \to X$. A set O in X is open if and only if $\psi^{-1}(O)$ is open in Z.

In order to show that X is an ANR we want to apply theorem 7.1, condition (a).

Let $\alpha = \{U_\lambda\}$ be an open covering of X. We consider the covering $\psi^{-1}(\alpha) = \{\psi^{-1}(U_\lambda)\}$ of Z. Making use of lemma 8.1, we can define a $\psi^{-1}(\alpha)$-deformation $k_t : Z \to Z$ such that

(4) $$k(z, 0) = z \quad \text{for } z \in Z,$$

(5) $$k(z, t) = z \quad \text{for } z \in A_1,\ t \in I,$$

(6) $$k(z, t) = z \quad \text{for } z \in X_2,\ t \in I,$$

(7) $$k(z, t) \in X_1 \quad \text{for } z \in X_1,\ t \in I,$$

(8) there is an open set V in Z, $A_1 \subset V \subset X_1$, for which $k_1(V) = A_1$.

Define

$$h_t = \psi\, k_t\, \psi^{-1} : X \to X.$$

Because of (5) and (6) h_t is single-valued. As in [8] we prove that $h(x, t)$ is continuous. Thus h_t is a deformation and clearly an α-deformation. It remains to be proved that $h_1 : X \to X$ has the property in condition (a) of theorem 7.1. This we prove in the following formulation. Let (Y, B) be any pair and $f : B \to X$ any mapping. Then $h_1 f : B \to X$ has a neighborhood extension. The proof will be rather similar to the proof of theorem 3.3, case a).

The two sets

$$F_1' = \psi(X_1 - V), \quad F_2' = \psi(X_2)$$

are disjoint closed subsets of X. Hence

$$F_1 = f^{-1}(F'_1),\ F_2 = f^{-1}(F'_2)$$

are two disjoint closed subsets of Y. Take two disjoint open sets Y_1 and Y_2 in Y such that

$$Y_1 \supset F_1,\ Y_2 \supset F_2.$$

Then $Y_0 = Y - (Y_1 \cup Y_2)$ is a closed subset of Y. Set $B_i = B \cap Y_i$, $i = 0, 1, 2$. We have

$$f(B_0) \subset \psi(V - A_1), \tag{9}$$

$$f(B_1) \subset \psi(X_1 - A_1),\ f(B_2) \subset \psi(V \cup X_2). \tag{10}$$

From (9) and (10) we obtain:

$$\psi^{-1} f \,|\, B_0 \text{ is single-valued, } k_1 \psi^{-1} f(B_0) \subset A_1, \tag{11}$$

$$\psi^{-1} f \,|\, B_1 \text{ is single-valued, } k_1 \psi^{-1} f(B_1) \subset X_1, \tag{12}$$

$$h_1 f(B_2) = \psi k \psi^{-1} f(B_2) \subset \psi(A_1 \cup X_2) = \psi(X_2). \tag{13}$$

We now study the mapping $k_1 \psi^{-1} f \,|\, B_0$, making use of (11). Since A_1 is an ANR, and B_0 is a closed subset of Y_0, there is a neighborhood extension $g'_0 : U_0 \to A_1$ of $k_1 \psi^{-1} f \,|\, B_0$ relative to A_1 to an open set U_0 in Y_0.

Consider the set $U_0 \cup B_1$, which is a closed subset of $U_0 \cup Y_1$. Because of (11) and (12) $k_1 \psi^{-1} f \,|\, B_0 \cup B_1$ is single-valued and takes values in X_1. Since $k_1 \psi^{-1} f \,|\, B_0 \cup B_1$ and $g'_0 : U_0 \to A_1$ agree on the intersection $B_0 = (B_0 \cup B_1) \cap U_0$, they define a function $l : U_0 \cup B_1 \to X_1$, which is continuous, since U_0 and $B_0 \cup B_1$ are closed in $U_0 \cup B_1$. Since X_1 is an ANR, there is an extension $g'_1 : U_1 \to X_1$ of l to an open neighborhood U_1 of $U_0 \cup B_1$ in $U_0 \cup Y_1$.

Thereafter, consider the closed set $U_0 \cup B_2$ in $U_0 \cup Y_2$. Using (13) we can define a mapping $m : U_0 \cup B_2 \to \psi(X_2)$ by

$$m(y) = \psi g'_0(y) \quad \text{for } y \in U_0,$$

$$m(y) = h_1 f(y) \quad \text{for } y \in B_0 \cup B_2.$$

Since for $y \in B_0$ we have

$$\psi g'_0(y) = \psi k_1 \psi^{-1} f(y) = h_1 f(y),$$

$m(y)$ is uniquely determined, and since U_0 and $B_0 \cup B_2$ are closed in $U_0 \cup B_2$, m is continuous. The space $\psi(X_2)$ is homeomorphic to X_2 and is therefore an ANR. Hence there is an extension $g_2 : U_2 \to \psi(X_2)$ of m to an open neighborhood U_2 of $U_0 \cup B_2$ in $U_0 \cup Y_2$.

Finally, put $U = U_1 \cup U_2$ and define $g : U \to X$ by

$$g(u) = \psi g'_1(u) \quad \text{for } u \in U_1,$$

$$g(u) = g_2(u) \quad \text{for } u \in U_2.$$

On $U_0 = U_1 \cap U_2$

$$\psi g_1'(u) = \psi g_0'(u) = m(u) = g_2(u).$$

Hence $g(u)$ is uniquely determined. As in the proof of theorem 3.3, case a) we see firstly that U_1 and U_2 are closed in U, showing that g is continuous, and secondly that U is open. Since $g \mid B = h_1 f$, g is therefore a neighborhood extension of $h_1 f$.

We can now apply theorem 7.1. For we have shown that X satisfies the condition (a) of theorem 7.1. Hence X is an ANR. This proves theorem 8.2.

Remark 8.3. The assumption that A_1 is compact is used in this proof only when showing that X is separable metric. When A_1 is non-compact, X is not necessarily metrizable. This is shown by taking X_1 to be the real line, A_1 the set of integers, and X_2 a single point.

REFERENCES. [1] **K. Borsuk,** Über eine Klasse von lokal zusammenhängenden Räumen, Fund. Math. *19,* 220—242 (1932). — [2] ——, Zur kombinatorischen Eigenschaften der Retrakte, Fund. Math. *21,* 91—98 (1933). — [3] ——, Sur un espace compact localement contractile qui n'est pas un rétracte absolu de voisinage, Fund. Math. *35,* 175—180 (1948). — [4] **R. H. Fox,** A characterization of absolute neighborhood retracts, Bull. Amer. Math. Soc. *48,* 271—275 (1942). — [5] **W. Hurewicz** and **H. Wallman,** Dimension theory, Princeton 1941. — [6] **S. Kaplan,** Homology properties of arbitrary subsets of Euclidean spaces, Trans. Amer. Math. Soc. *62,* 248—271 (1947). — [7] **C. Kuratowski,** Sur les espaces localement connexes et péaniens en dimension *n,* Fund. Math. *24,* 269—287 (1935). — [8] **J. H. C. Whitehead,** Note on a theorem due to Borsuk, Bull. Amer. Math. Soc. *54,* 1125—1132 (1948). — [9] **M. Wojdysławski,** Rétractes absolus et hyperespaces des continus, Fund. Math. *32,* 184—192 (1939). — [10] **T. Yajima,** On a local property of absolute neighbourhood retracts, Osaka Math. Journal *2,* 59—62 (1950).

Tryckt den 28 februari 1951

Uppsala 1951. Almqvist & Wiksells Boktryckeri AB