

$$\delta_1\omega = \omega^{\rho_1+1}, \text{ and } \delta_1j = \omega^{\rho_1}v_1j + \omega^{\rho_2}v_2 + \cdots + \omega^{\rho_z}v_z < \omega^{\rho_1}(v_1j+1);$$

$$\sigma(\delta_1\mu, \delta_1j) < \sigma(\delta_1\mu, \omega^{\rho_1}(v_1j+1)) < \delta_1\mu + \omega^{\rho_1+1} = \delta_1\mu + \delta_1\omega.$$

By (2),  $\pi(\delta^\mu, \delta^j) < \omega^{\delta_1\mu + \delta_1j} = (\omega^{\delta_1})^{(\mu+j)} \leq \delta^{\mu+j} \leq \delta^\delta$ .

Hence by (1), the order type of  $S$  is less than  $\pi(\omega^\delta, \delta^\delta)$ . This is a contradiction since  $S$  was the segment of  $M^\delta$  of order type  $\pi(\omega^\delta, \delta^\delta)$ .

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## A CHARACTERIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS

RALPH H. FOX

By an *absolute neighborhood retract* (ANR) I mean a separable metrizable space which is a neighborhood retract of every separable metrizable space which contains it and in which it is closed. This generalization of Borsuk's original definition<sup>1</sup> was given by Kuratowski<sup>2</sup> for the purpose of enlarging the class of absolute neighborhood retracts to include certain spaces which are not compact. The space originally designated by Borsuk as absolute neighborhood retracts (or  $\mathfrak{R}$ -sets) will now be referred to as compact absolute neighborhood retracts. Many of the properties of compact ANR-sets hold equally for the more general ANR-sets.<sup>3</sup>

The Hilbert parallelotope  $Q$ , that is, the product of the closed unit interval  $[0, 1]$  with itself a countable number of times is a "universal" compact ANR in the sense that<sup>4</sup> every compact ANR is homeomorphic to a neighborhood retract of  $Q$ . The classical theory of Borsuk makes good use of the imbedding of compact ANR-sets in  $Q$ . The problem solved here is that of finding a "universal" ANR.

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<sup>1</sup> Fundamenta Mathematicae, vol. 19 (1932), pp. 220-242.

<sup>2</sup> Fundamenta Mathematicae, vol. 24 (1935), p. 270, Footnote 1.

<sup>3</sup> Ibid., pp. 272, 276, and 277, and Footnote 1, p. 279 and Footnote 3. Note that Theorem 12, Fundamenta Mathematicae, vol. 19 (1932), p. 229, is not true for general ANR-sets. In fact let  $A = \sum S_n$  where  $S_n$  is the plane circle of radius  $2^{-n}$  and center  $(3 \cdot 2^{-n}, 0)$ ; let  $f(x, y) = (x, |y|)$  for  $(x, y) \in A$  and let

$$f_n(x, y) = \begin{cases} (x, |y|), & \text{for } (x, y) \in A - S_n, \\ (x, y), & \text{for } (x, y) \in S_n. \end{cases}$$

Then  $f_n \rightarrow f$  in  $A^4$ ;  $f$  can be extended to the half-plane  $\{x > 0\}$ , but none of the maps  $f_n$  can.  $A$  is an ANR-set. Theorem 16, Fundamenta Mathematicae, vol. 19 (1932), p. 230, is also false for general ANR-sets.

<sup>4</sup> Fundamenta Mathematicae, vol. 19 (1932), p. 223.

Strictly speaking, the problem as just stated has no solution; there is no single "universal" ANR, but rather a whole class of ANR-sets which together serve in the "universal" capacity. Such a class of ANR-sets is the collection of subsets of the Hilbert parallelotope  $Q \times [0, 1]$  which contains the open subset<sup>5</sup>  $Q \times (0, 1]$  of  $Q \times [0, 1]$ .

**THEOREM 1.** *For a separable metrizable space  $X$  the following three conditions are equivalent:*

- (1)  $X$  is an ANR-set;
- (2) There is a homeomorphism  $f$  of  $X$  into  $Q$  such that  $f(X) \times [0]$  is a neighborhood retract of  $f(X) \times [0] + Q \times (0, 1]$ ;
- (3)  $f(X) \times [0]$  is a neighborhood retract of  $f(X) \times [0] + Q \times (0, 1]$  for every homeomorphism  $f$  of  $X$  into  $Q$ .

(1) $\rightarrow$ (3): If  $f$  is a homeomorphism of an ANR-set  $X$  into  $Q$  then  $f(X) \times [0]$  is an ANR-set. Since  $Q$  is compact, so that

$$\overline{f(X) \times [0]} \subset Q \times [0],$$

it follows that  $f(X) \times [0]$  is closed in  $f(X) \times [0] + Q \times (0, 1]$ . Hence  $f(X) \times [0]$  is a neighborhood retract of  $f(X) \times [0] + Q \times (0, 1]$ .

(3) $\rightarrow$ (2): Since  $X$  is separable and metrizable a homeomorphism  $f$  exists by Urysohn's theorem.<sup>6</sup>

(2) $\rightarrow$ (1): Let  $M$  be a separable metrizable space containing  $X$  in which  $X$  is closed and let  $f$  be a homeomorphism of  $X$  into  $Q$ . By Tietze's theorem<sup>7</sup> there exists a continuous function  $g$  defined on  $M$  with values in  $Q$  such that  $g(x) = f(x)$  for every  $x \in X$ . Let  $M$  be metrized, with metric  $d$ , and let  $\rho(x) = \min \{1, d(x, X)\}$  for every  $x \in M$ . Let  $h(x) = (g(x), \rho(x))$ , so that  $h$  is a continuous function defined on  $M$  with values in  $f(X) \times [0] + Q \times (0, 1]$  which has the property  $h(M - X) \subset Q \times (0, 1]$ . Let  $V$  be a neighborhood of  $f(X) \times [0]$  in  $f(X) \times [0] + Q \times (0, 1]$  and let  $U = h^{-1}(V)$  so that  $U$  is a neighborhood of  $X$  in  $M$ . If  $r$  is a retraction of  $V$  onto  $f(X) \times [0]$  then the mapping<sup>8</sup>  $f^{-1}\pi r h|_U$ , where  $\pi$  denotes the projection of  $Q \times [0]$  onto  $Q$ , is a retraction of  $U$  onto  $X$ .

Kuratowski also gave an analogous generalization of the notion of absolute retract.<sup>2</sup> According to the extended definition a separable metrizable space is an *absolute retract* (AR) if it is a retract of every containing separable metrizable space in which it is closed.

<sup>5</sup> The symbol  $(0, 1]$  denotes the half-open interval  $0 < t \leq 1$ .

<sup>6</sup> Alexandroff and Hopf, *Topologie*, p. 81.

<sup>7</sup> *Ibid.*, p. 73.

<sup>8</sup> If  $B \subset B'$  and  $e$  is a function defined on  $B'$  then the notation  $d = e|_B$  means that  $d$  is the function defined on  $B$  such that  $d(x) = e(x)$  for every  $x \in B$ .

**THEOREM 1'.** *For a separable metrizable space  $X$  the following three conditions are equivalent:*

- (1')  $X$  is an AR;
- (2') There is a homeomorphism  $f$  of  $X$  into  $Q$  such that  $f(X) \times [0]$  is a retract of  $f(X) \times [0] + Q \times (0, 1]$ ;
- (3')  $f(X) \times [0]$  is a retract of  $f(X) \times [0] + Q \times (0, 1]$  for every homeomorphism  $f$  of  $X$  into  $Q$ .

The proof of this theorem is an obvious modification of the preceding proof.

**COROLLARY.** *If  $C$  denotes the open  $n$ -cell  $0 < x_i < 1$  ( $i = 1, \dots, n$ ) and  $D$  denotes the closed  $n$ -cell  $0 \leq x_i \leq 1$  ( $i = 1, \dots, n$ ) then any set  $E$  such that  $C \subset E \subset D$  is an AR.*

By condition (2') and a retraction of  $Q \times [0, 1]$  onto  $D \times [0, 1]$  it is sufficient to show that  $E \times [0]$  is a retract of  $E \times [0] + D \times (0, 1]$ . This can be done by projecting from the point  $(1/2, \dots, 1/2, -1)$  of Euclidean  $(n+1)$ -space.

It may be worth noting that conditions (2) and (2') make possible a simpler proof of the Borsuk-Kuratowski<sup>9</sup> theorem(s):

*If  $W$  is a closed subset of a normal space  $Z$  and  $X$  is an AR-set (ANR-set) then every continuous map of  $W$  into  $X$  can be extended to  $Z$  (to a neighborhood of  $W$  in  $Z$ ).*

In fact conditions (2) and (2') replace a theorem of Kuratowski<sup>10</sup> which involves infinite polyhedra.

**THEOREM 2.** *An ANR is locally contractible.<sup>11</sup> An AR is also contractible.*

Using (2) we can suppose that our ANR-set  $Y$  is contained in  $Q \times [0]$  and that there is a retraction  $r$  of an open neighborhood  $V$  of  $Y$  in  $Y + Q \times (0, 1]$  onto  $Y$ . But  $V$  is the intersection of  $Y + Q \times (0, 1]$  with an open set  $V'$  of  $Q \times [0, 1]$ . Let  $y \in Y$  and let  $S_\epsilon$  denote the  $\epsilon$ -sphere in  $Q \times [0, 1]$  about the point  $y$ . Since  $r$  is continuous there is a  $\delta > 0$  such that the intersection  $T_\delta$  of the  $\delta$ -sphere  $S_\delta$  and  $Y + Q \times (0, 1]$  is contained in  $V'$ , hence in  $V$ , and  $r(T_\delta) \subset S_\epsilon$ . Let  $u_t$  denote a contraction of  $S_\delta$  to a point  $p \in S_\delta \cdot (Q \times (0, 1])$  which moves points rectilinearly, so that  $u_t(x) \in Q \times (0, 1]$  for every  $0 < t \leq 1$  and

<sup>9</sup> Fundamenta Mathematicae, vol. 24 (1935), p. 275.

<sup>10</sup> Fundamenta Mathematicae, vol. 24 (1935), p. 266, Theorem 2.

<sup>11</sup> But not uniformly. See the example in Footnote 3. This theorem was proved by Borsuk, Fundamenta Mathematicae, vol. 19 (1932), p. 237 for compact ANR-sets.

$y \in Y \cdot S_\delta$ . Then  $ru_t| Y \cdot S_\delta$  contracts  $Y \cdot S_\delta$  in  $Y \cdot S_\epsilon$ . The second statement is a consequence of Theorem 3'.

**THEOREM 3.** *A separable metrizable space  $X$  is an ANR if and only if for every separable metrizable space  $M$  containing  $X$  (in which  $X$  need not be closed!) there is a neighborhood  $U$  of  $X$  and a continuous function  $h$  defined on  $X \times [0] + U \times (0, 1]$  with values in  $X$  such that<sup>8</sup>  $h|X \times [0, 1]$  is a deformation.<sup>12</sup>*

Suppose  $X$  is an ANR and  $M$  a separable metrizable space containing  $X$ . We may assume that  $M \subset Q$ . By (2) and (3) there is an open neighborhood  $V'$  of  $X \times [0] + Q \times (0, 1]$  and a retraction  $r$  of  $V = V' \cdot (X \times [0] + Q \times (0, 1])$  onto  $X \times [0]$ . Let  $\lambda(x) = d(x \times [0], Q \times [0, 1] - V')$  for every  $x \in M$  and let  $U = \pi(V' \cdot (Q \times [0]))$  where, as before,  $\pi$  denotes the projection of  $Q \times [0]$  onto  $Q$ . Define for every  $(x, t) \in X \times [0] + U \times (0, 1]$ ,

$$\begin{aligned} h(x, t) &= \pi r(x, t), && \text{when } t \leq \lambda(x), \\ &= \pi r(x, \lambda(x)), && \text{when } t \geq \lambda(x). \end{aligned}$$

Since  $\lambda$  is continuous and  $\lambda(x) > 0$  when  $x \in U$  it follows that  $h$  is continuous.

Conversely, let  $U$  be a neighborhood of  $X$  in  $M = Q$  and let  $h$  be a continuous function defined on  $X \times [0] + U \times (0, 1]$  with values in  $X$  such that  $h(x, 0) = x$  for every  $x \in X$ . Then  $h$  is a retraction of  $X \times [0] + U \times (0, 1]$  onto  $X \times [0]$ . Furthermore  $X \times [0] + U \times (0, 1]$  is a neighborhood of  $X \times [0]$  in  $X \times [0] + Q \times (0, 1]$ .

**THEOREM 3'.** *A separable metrizable space  $X$  is an AR if and only if for any separable metrizable space  $M$  containing  $X$  there is a continuous function  $h$  defined on  $X \times [0] + M \times (0, 1]$  with values in  $X$  such that<sup>8</sup>  $h|X \times [0, 1]$  is a contraction.<sup>12</sup>*

Let  $X$  be an AR and  $M$  a separable metrizable space containing  $X$ ; we may assume that  $M \subset Q$ . Let  $r$  be a retraction of  $X \times [0] + Q \times (0, 1]$  onto  $X \times [0]$ . Let  $p \in Q$  and let

$$h(x, t) = \pi r(tp + (1 - t)x, t)$$

for every  $(x, t) \in X \times [0] + M \times (0, 1]$ , where  $\pi$  is the projection of  $Q \times [0]$  onto  $Q$ . Then  $h$  maps  $X \times [0] + M \times (0, 1]$  continuously into  $X$  and  $h|X \times [0, 1]$  is a contraction of  $X$ .

The converse is proved as in Theorem 3.

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<sup>12</sup> A deformation of  $X$  is a continuous mapping  $h$  of  $X \times [0, 1]$  into  $X$  such that  $h(x, 0) = x$  for every  $x \in X$ . If  $h(X, 1)$  is a point then  $h$  is called a contraction of  $X$ .

If  $X$  is locally compact the deformation  $h|X \times [0, 1]$  of Theorems 3 and 3' can be chosen in advance of  $M$ . For then there exists<sup>13</sup> a compact set  $M^*$  and a homeomorphism  $g$  of  $X$  into  $M^*$  such that  $M^* - g(X)$  is a point. (We can suppose  $X$  not compact so that  $M^* \neq g(X)$ .) Let  $M^* \subset Q$ . The homeomorphism  $g$  can be extended<sup>13</sup> to a continuous mapping  $g^*$  of  $\bar{X}$  into  $M^*$  by defining  $g^*(\bar{X} - X) = M^* - g(X)$ . The mapping  $g^*$  of  $\bar{X}$  into  $Q$  can be extended, by Tietze's theorem, to a mapping  $k$  of  $M$  into  $Q$ . In the case of Theorem 3 let  $h$  be the mapping of  $X \times [0] + U \times (0, 1]$  into  $X$  defined by

$$h(x, t) = g^{-1}\pi r(k(x), \min \{t, \lambda(x)\}),$$

where  $U = g^{-1}\pi(V' \cdot (Q \times [0]))$ . In the case of Theorem 3' let  $h$  be the mapping of  $X \times [0] + M \times (0, 1]$  into  $X$  defined by

$$h(x, t) = g^{-1}\pi r(tp + (1 - t)k(x), t).$$

In both cases  $h|X \times [0, 1]$  is independent of  $M$ .

If  $X$  is not locally compact it may not be possible to pick a deformation  $h|X \times [0, 1]$  satisfying the conditions of Theorems 3 or 3' for all  $M$ . An example is the AR-set  $\{0 \leq x \leq 1; y = 0\} + \sum_{n=1}^{\infty} \{x = 1/n; 0 \leq y \leq 1\}$ .

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<sup>13</sup> Alexandroff and Hopf, *Topologie*, I, p. 93.