ANNALS OF MATHEMATICS
Vol. 51, No. 3, May, 1950

SOME CHARACTERIZATIONS OF INTERIOR MAPS

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(Received March 9, 1949)

The purpose of this note is to characterize light interior maps in several ways. Each of the characterizations emphasizes the one-dimensional nature of these maps. In the first section we characterize by means of null-homotopic maps into the unit circle and also by one-dimensional cohomology groups the quasi-monotone maps introduced by Wallace [4]. Since light interior maps coincide with light quasi-monotone maps, we also obtain characterizations of the former. In the last section we generalize a theorem of Whyburn [5, p. 186] to prove that a light map f of X onto Y is interior if and only if every map h of the unit interval I into Y can be factored as fg with $g:I \to X$, where $g(0)$ is an arbitrary point of $f^{-1}h(0)$.

1. Homotopy and homology characterizations

In this section we use the method of null-homotopic maps into the unit circle in a fashion similar to that used by Eilenberg [2, pp. 174–175] to obtain two characterizations. According to Wallace [4], a map f of X onto Y, X and Y locally connected compacta, is quasi-monotone if and only if V a region in Y and U a component of $f^{-1}(V)$ implies that $f(U) = V$. We recall that if f is light (i.e., each $f^{-1}(y)$, $y \in Y$, is 0-dimensional), then f is quasi-monotone if and only if it is interior.

Let X, Y be topological spaces and A, B closed subsets of X, Y respectively. As usual, we say that a mapping f of X into Y maps the pair (X, A) into the pair (Y, B) if $f(A) \subset B$. Moreover two maps $f, g: (X, A) \to (Y, B)$ are said to be homotopic if and only if there exists a homotopy $h(x, t)$ connecting f and g such that $h(A, t) \subset B$ for each t. If (X, A) is a pair with X compact, then $H^1(X, A)$ will denote the one-dimensional Čech cohomology group of X relative to A with integer coefficients.

We indicate first a slight variation of Bruschlinsky's Theorem as given by Dowker [1, p. 226].

LEMMA 1. *Let X be a compact Hausdorff space and A a closed subset of X. Consider the family Φ of maps $g:(X, A) \to (S_1, x)$, where S_1 is the unit circle and $x \in S_1$. Let u be a generator of $H^1(S_1, x)$. Then the correspondence between $g \in \Phi$ and g^*u induces a 1–1 correspondence between $H^1(X, A)$ and the homotopy classes of Φ.*

PROOF. Define X' to be the space obtained from the decomposition of X which has as elements the points of $X - A$ together with the element A. Denote by A' the point generated by A and denote by $f: (X, A) \to (X', A')$ the map associated with the decomposition. Then $f^*:H^1(X', A') \to H^1(X, A)$ is an isomorphism onto, as is well-known. Moreover, the inclusion map $i:X' \to (X', A')$

generates an isomorphism onto, $i^*:H^1(X', A') \to H^1(X')$, since A' is a single point. It is known that there is a 1–1 correspondence between homotopy classes of maps $g:(X', A') \to (S_1, x)$ and homotopy classes of maps $g:X' \to S_1$. Let v be a generator of $H^1(S_1)$. It follows from Dowker's theorem that the homotopy classes of maps $g:X' \to S_1$ are in 1–1 correspondence with the elements $g^*v \in H^1(X')$. The lemma follows readily from the consideration of the isomorphisms f^* and i^*, together with the natural isomorphism of $H^1(S_1, x)$ onto $H^1(S_1)$. We note that the lemma would also follow for spaces other than compact spaces if it were known that f^* is an isomorphism onto.

THEOREM 1. *Let X and Y be locally connected, connected compacta, and let f be a map of X onto Y. Then the following are equivalent:*

(1) *f is quasi-monotone;*

(2) *If V is a region (an open connected set) in Y and U a component of $f^{-1}(V)$, and $g:(\bar{V}, \bar{V} - V) \to (S_1, x)$, where S_1 is the unit circle and $x \in S_1$, then $g:(\bar{V}, \bar{V} - V) \to (S_1, x)$ is null-homotopic if and only if $gf:(\bar{U}, \bar{U} - U) \to (S_1, x)$ is null-homotopic;*

(3) *If V is a region in Y and U a component of $f^{-1}(V)$, then $f^*:H^1(\bar{V}, \bar{V} - V) \to H^1(\bar{U}, \bar{U} - U)$ is an isomorphism into, where f^* indicates the homomorphism induced by f.*

PROOF. We prove (1) and (2) equivalent, and (2) and (3) equivalent. We suppose first that f is quasi-monotone and let V be a region in Y and U a component of $f^{-1}(V)$. Consider a map $g:(\bar{V}, \bar{V} - V) \to (S_1, x)$ such that $gf:(\bar{U}, \bar{U} - U) \to (S_1, x)$ is homotopic to the constant map into x. We suppose that $x = 1$. Since gf is null-homotopic, there exists [2, p. 162] a map ϕ of $\bar{U}$ into the reals such that $\theta\phi = gf \mid \bar{U}$, where θ denotes the map defined by $\theta(x) = e^{2\pi i x}$, and such that $\phi(\bar{U} - U)$ is a single integer n. Define a map ψ of $\bar{V}$ into the reals by $\psi(x) = \inf \phi(f^{-1}(x) \cap \bar{U})$. It follows that $\theta\psi = g$ and that $\psi(\bar{V} - V)$ is the integer n. To show that g is null-homotopic we have only to show that ψ is continuous. Since $\phi(f^{-1}(x) \cap \bar{U})$, $x \in \bar{V} - V$, consists of a single point, it follows that ψ is continuous at each point of $\bar{V} - V$. It also follows that ψ is lower semi-continuous, since f and ϕ are continuous. Hence we have only to show that ψ is upper semi-continuous at each point of V.

Let $y \in V$ and let W be a neighborhood of $\psi(y)$ small enough so that $\theta \mid \overline{W}$ is a homeomorphism. Since $f^{-1}(y) \cap \bar{U}$ is compact, there exists an $x \in \bar{U}$ such that $\phi(x) = \psi(y)$. Since $f^{-1}(y) \cap \bar{U} = f^{-1}(y) \cap U$, we have that $x \in U$. Let P be a region in V containing y and such that $g(P) \subset \theta(W)$. Denote by Q the component of $f^{-1}(P)$ which contains x, and note that since $f(\bar{U} - U) \subset \bar{V} - V$ we have $Q \subset U$. We then have that $gf(Q) = g(P)$ is contained in the interior of $\theta(W)$. But $\phi(Q)$ is contained in W for otherwise we would not have $g(P) \subset \theta(W)$. We have that Q contains points of each $f^{-1}(p)$, $p \in P$, by the quasi-monotone property of f. Then $\psi(p)$, $p \in P$, is a number $\leqq \max (t \mid t \in W)$. Hence ψ is upper semi-continuous at y, and is then continuous. The factorization $g = \theta\psi$ implies that g is null-homotopic and (1) implies (2).

We now show that (2) implies (1). Suppose that f is not quasi-monotone. There is then a region V' in Y and a component U of $f^{-1}(V')$ such that $f(U) \neq V'$. It is clear that $U \neq X$, since otherwise f would not map X onto Y. Then, since X is connected, we have that $\bar{U} - U \neq 0$. Let $x \in V' - f(U)$. Since $f(U)$ is closed in V', we may select a neighborhood W of x such that $\bar{W} \subset V'$ and $\bar{W} \cap f(U) = 0$. Consider the component V of $V' - \bar{W}$ which contains $f(U)$. It is clear that V exists since $f(U)$ is connected. We have that $\bar{V} - V \subset (\bar{V}' - V') \cup (\bar{W} - W)$. Moreover $C = (\bar{V} - V) \cap (\bar{V}' - V')$ contains $f(\bar{U} - U)$ and is then not empty. Also $D = (\bar{V} - V) \cap (\bar{W} - W) \neq 0$ since if $y \in V$ then y may be joined to x by an arc in U. The first point of this arc in the order y, x on $\bar{W}$ will be a point of D. We note also that U is a component of $f^{-1}(V)$.

Define a map g' of $\bar{V}$ onto the unit interval as follows: $g': \bar{V} \to [0, 1]$, $g'^{-1}(0) = C$, $g'^{-1}(1) = D$. The existence of g' follows from Urysohn's Lemma. The map g' defines a map $g = \theta g'$ from $(\bar{V}, \bar{V} - V)$ into $(S_1, 1)$. We note that $gf(\bar{U})$ is a proper subset of S_1, since $g'f(\bar{U})$ does not contain 1 and is contained in $[0, 1]$. It follows that $gf: (\bar{U}, \bar{U} - U) \to (S_1, 1)$ is null-homotopic. We show that g is not homotopic to a constant, which will show that (2) implies (1). Suppose g is homotopic to a constant. There exists, then, a factorization $g = \theta\psi$ where ψ maps $\bar{V}$ into the reals with $\psi(\bar{V} - V)$ a single integer n, and $\theta(x) = e^{2\pi ix}$. Since $g^{-1}(1) = \bar{V} - V$ it follows that $\psi(\bar{V})$ contains but a single integer. Moreover, $\psi(\bar{V})$ is compact and connected, and $\psi(V)$ is connected and contains no integer. Then $\psi(\bar{V})$ is a proper subset of some half-open interval from an integer k to $k + 1$. Hence $\theta\psi(\bar{V})$ is a proper subset of S_1. This is absurd, since $g'(\bar{V}) = [0, 1]$. Then g is not null-homotopic, and (2) implies (1).

That (2) is equivalent to (3) follows easily from Lemma 1. We prove that (2) implies (3), the other part being quite similar. Consider the homomorphism $f^*: H^1(\bar{V}, \bar{V} - V) \to H^1(\bar{U}, \bar{U} - U)$. Suppose $u \in H^1(\bar{V}, \bar{V} - V)$ is such that $f^*u = 0$. By Lemma 1, there exists $g: (\bar{V}, \bar{V} - V) \to (S_1, x)$ such that if v denotes the generator of $H^1(S_1, x)$, then $g^*v = u$. Consider $(gf)^*: H^1(S_1, x) \to H^1(\bar{U}, \bar{U} - U)$. We have $(gf)^*v = f^*g^*v = 0$. Hence gf is null-homotopic. But by the hypotheses of (2), g is then null-homotopic. Hence $u = 0$, and f^* is an isomorphism into.

Corollary. *Let X and Y be locally connected, connected compacta, and let f be a light map of X onto Y. Then the following conditions are equivalent*:

(1) *f is interior*;

(2) *condition* (2) *of Theorem* 1;

(3) *condition* (3) *of Theorem* 1.

Proof. The proof is an immediate consequence of Theorem 1 and the fact that a light map is quasi-monotone if and only if it is interior.

2. A factorization theorem

The following theorem is a generalization of theorems of Stoïlow [3, p. 109] and Whyburn [5, pp. 186–187].

THEOREM 2. *Let X and Y be compacta and let f be a light interior map of A onto Y. Consider a map $h:I \to Y$ where I is the unit interval. Then for each $x \in f^{-1}h(0)$ there exists a map $g:I \to X$ such that $h = fg$ and $g(0) = x$.*

PROOF. Let $[e_i]$ be a sequence of positive numbers tending to 0. We define for each n a barycentric subdivision Σ_n of I such that for each l-simplex $\sigma^n \in \Sigma_n$ there corresponds a set C^n in X with the following properties:

1) C^n is a component of $f^{-1}h(\sigma^n)$ of diameter $<e_n$;
2) $\sigma_1^n \cap \sigma_2^n \neq 0$ implies $C_1^n \cap C_2^n \neq 0$;
3) Σ_{n+1} is a repeated barycentric subdivision of Σ_n ;
4) if $0 \in \sigma^n$ then $x \in C^n$.

We define Σ_n , $[C^n]$ inductively as follows. Suppose Σ_{k-1} , $[C^{k-1}]$ have been defined. There exists a $\delta > 0$ such that if M is a continuum in Y of diameter $<\delta$, then each component of $f^{-1}(M)$ is of diameter $<e_k$ and maps onto M [5, p. 131, p. 148]. We take for Σ_k a repeated barycentric subdivision of Σ_{k-1} such that if $\sigma^k \in \Sigma_k$, then diameter $h(\sigma^k) < \delta$. Order the 1-simplexes of Σ_k linearly from 0 to 1 as $\sigma_1^k , \cdots , \sigma_r^k$. Pick for C_1^k the component of $f^{-1}h(\sigma_1^k)$ which contains x. If C_{n-1}^k has been defined, pick for C_n^k any component of $f^{-1}h(\sigma_n^k)$ which intersects C_{n-1}^k . This is possible since each component of $f^{-1}h(\sigma_{n-1}^k)$, in particular C_{n-1}^k, maps onto $h(\sigma_{n-1}^k)$. Then Σ_k, $[C^k]$ clearly satisfy the conditions. Moreover, it is clear that the first stage in the induction is carried out in a similar fashion.

Suppose σ^n is a fixed 1-simplex of Σ_n . For $m \geqq n$, define $C^{n,m}$ to be the union of all sets C^m such that $\sigma^m \subset \sigma^n$ and $\sigma^m \in \Sigma_m$. It is important to note that $C^{n,m}$ is connected and that $f(C^{n,m}) = h(\sigma^n)$. We may use the diagonal process to pick a subsequence of $\Sigma_1 , \Sigma_2 , \cdots$, which we suppose the same as the original, with the following property: if $\sigma^n \in \Sigma_n$, then there exists $\lim_{(m)} C^{n,m}$. Define $D^n = \lim_{(m)} C^{n,m}$. Since each D^n is a continuum of X with diameter $D^n = \lim$ diameter $C^{n,m}$, it follows that diameter $D^n \leqq e_n$ since each $C^{n,m}$ has the same property. Also if $\sigma_1^n \cap \sigma_2^n \neq 0$ then $D_1^n \cap D_2^n \neq 0$, since for $m \geqq n$ we have $C_1^{n,m} \cap C_2^{n,m} \neq 0$.

Let $p \in I$, and for each n let σ_1^n , σ_2^n be the 1-simplexes of Σ_n which contain p (we allow $\sigma_1^n = \sigma_2^n$). Then $[D_1^n \cup D_2^n]$ is a decreasing sequence of sets in X whose diameters tend to 0. Define $g(p) = \bigcap_n (D_1^n \cup D_2^n)$. Then g is clearly single-valued and moreover $fg = h$. Let q be an interior point of $\sigma_1^n \cup \sigma_2^n$. Then clearly $g(q) \subset D_1^n \cup D_2^n$, which is of diameter $\leqq 2e_n$. This proves the continuity of g.

THEOREM 3. *Let X and Y be locally connected compacta and let $f:X \to Y$ be a light mapping of X onto Y. Then a necessary and sufficient condition that f be interior is that for each map $h:I \to Y$ and each $x \in f^{-1}h(0)$ there exists a map $g:I \to X$ with $fg = h$ and $g(0) = x$.*

PROOF. The necessity follows from Theorem 2. Suppose that f is not interior. There is, then, a point $y \in Y$ and a point $x \in f^{-1}(y)$ such that if U is any sufficiently small neighborhood of x, then x is not an interior point of $f(U)$. Since f is light, for a fixed $\varepsilon > 0$ there is a $\delta > 0$ such that if C is any continuum in Y of diameter $<\delta$, then each component of $f^{-1}(C)$ is of diameter $< \varepsilon$ [5, p. 161]. There is, since Y is locally connected, a $\sigma > 0$ such that if $z \in N_\sigma(y)$ (i.e., the spherical σ-neighborhood about y), then z can be joined to y by an arc I_z of diameter $<\delta$. Let $z \in N_\sigma(y)$ be such that $z \notin f(N_\varepsilon(x))$. Let $h: I \to I_z$ map I topologically onto I_z with $h(0) = y$. Let $g: I \to X$ be such that $fg = h$ and $g(0) = x$. Then $g(I)$ is contained in a component of $f^{-1}h(I)$ and hence is of diameter $<\varepsilon$. Then $g(I) \subset N_\varepsilon(x)$. This, however, is a contradiction since $g(1) \in N_\varepsilon(x)$ is such that $fg(1) = h(1) = z$. It follows that f is interior.

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