ON UNIQUENESS OF END-SUMS AND 1-HANDLES AT INFINITY

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PRELIMINARY

Abstract.

1. Introduction

Since the early days of topology, it has been useful to combine spaces by simple gluing operations. The connected sum operation for closed manifolds has roots in nineteenth century surface theory. Its cousin, the boundary sum of compact manifolds with boundary, is also a classical operation. Both of these operations are well understood. In the setting of oriented manifolds, for example, the connected sum of two connected manifolds is unique, as is the boundary sum of two manifolds with connected boundary.

The analogue for open manifolds of the boundary sum is called the *end sum*. While this operation has been used for manifolds of various dimension since the 1980s, it is less well known and understood. In contrast with boundary sums, end sums of one-ended oriented manifolds need not be uniquely determined, even up to proper homotopy [CH14]. In fact, the uniqueness issue is quite subtle, and is the primary topic of this paper. We present examples in various categories (homotopy, TOP, PL, and DIFF) where uniqueness fails, but the failure cannot be detected in weaker categories.

In counterpoint, we show that under reasonable hypotheses the operation *is* unique in all categories and apply this result to exotic smoothings of open 4-manifolds. We put our results into a broader context. Just as boundary summing is a special case of attaching a 1-handle, end-summing is a case of *attaching a 1-handle at infinity.* We obtain general results about uniqueness of attaching collections of 0- and 1-handles at infinity. We conclude that end sums, and more generally, collections of handles at infinity with index at most one, can be controlled in broad circumstances, although deep questions remain.

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End summing is the natural analogue of boundary summing. For the latter, we choose codimension-0 embeddings of a disk into the boundaries of the two summands, then use these to attach a 1-handle. For an end sum of open manifolds, we attach a 1-handle at infinity, guided by a properly embedded ray in each summand. Informally, we can think of the 1-handle at infinity as a piece of tape joining the two manifolds. This is made precise in Definition 2.1 below. In particular, boundary-summing two compact manifolds has the effect of end-summing their interiors.

While the end sum notion seems obvious, the authors have been unable to find explicit appearences of it before the second author's 1983 paper [G83]. The germ of the idea is present in Mazur's 1959 paper [M59] and Stallings' 1965 paper [St65]. In [G83], the operation was used to construct a new (unoriented) diffeomorphism type of exotic $\mathbb{R}^4$, shortly following the emergence of the first exotic $\mathbb{R}^4$ when theorems of Donaldson and Freedman completed a seminal construction of Casson. Subsequently, [G85] constructed infinitely many exotic $\mathbb{R}^4$'s and gave a more systematic treatment of the end-sum operation on the space $\mathcal{R}$ of oriented diffeomeorphism types homeomorphic to $\mathbb{R}^4$. In particular, infinite end-sums were shown to be well-defined and independent of order and grouping, implying that the resulting monoid structure on $\mathcal{R}$ has no inverses.

Since that time, end-summing with exotic $\mathbb{R}^4$s has become a standard technique for constructing many exotic smoothings on a given open 4-manifold. End-summing has also been used in other dimensions, for example by Ancel (unpublished) in the 1980s to study high-dimensional Davis manifolds, and by Tinsley and Wright (1997) [TW97] and Myers (1999) [My99] to study 3-manifolds. In 2012, the first author, with King and Siebenmann, gave a general treatment [CKS12] of end sum (called *CSI, connected sum at infinity*, therein) in all dimensions and categories (TOP, PL, and DIFF). As a corollary, there resulted a classification of multiple hyperplanes in $\mathbb{R}^n$ for all $n \neq 3$. The resulting classification was recently used by Belegradek [B14] to study certain interesting open aspherical manifolds.

While [G85] showed that end sums are uniquely determined for manifolds in $\mathcal{R}$, the general case is more complex. First, boundary summing can already fail to be unique for simple reasons: if a summand has disconnected boundary, then we must specify which boundary component to use. For example, nondiffeomorphic boundary components can lead to boundary sums with nondiffeomorphic boundaries. We must also be careful to specify orientations — a pair of disk bundes over S^2 with nonzero Euler numbers can be boundary summed in two different ways, distinguished by their signatures (0 or ± 2). In general, we should specify an orientation on each orientable boundary component receiving a 1-handle.

Similarly, for end sums and 1-handles at infinity, we must specify which ends of the summands we are using and an orientation on each such end (if orientable). Unlike the compact case, however, this information may still be insufficient for specifying the resulting diffeomorphism type. One difficulty is specific to dimension 3: the rays in use can be knotted. Myers [My99] showed that uncountably many homeomorphism types of contractible manifolds can be obtained by end-summing two copies of $\mathbb{R}^3$ along knotted rays. For this reason, the present paper focuses on dimensions above 3. However, another difficulty persists in high dimensions: rays determining a given end need not be properly homotopic. The first author and Haggerty [CH14] constructed examples of pairs of one-ended oriented n-manifolds ($n \geq 4$) that are connected at infinity but can be summed in different ways that are not even properly homotopy equivalent. We explore this phenomenon more deeply in Section 3. After sketching the key example of [CH14], we exhibit more subtle examples of nonuniqueness of end-summing (and related constructions) on fixed oriented ends. Example 3.3 gives topological 5-manifolds with properly homotopy equivalent but nonhomeomorphic end sums, and PL n-manifolds ($n \geq 9$) whose end sums are properly homotopy equivalent but not PL-homeomorphic. Example 3.4 gives end sums of smooth manifolds ($n \geq 8$) that are PL-homeomorphic but not diffeomeorphic. In dimension 4, the same construction gives smooth manifolds whose end sums are naturally identified in the topological category, but whose smoothings are not stably isotopic. (Distinguishing their diffeomorphism types seems difficult.)

These failures of uniqueness arise from complicated fundamental group behavior at the relevant ends, contrasting with uniqueness associated with the simply connected end of $\mathbb{R}^4$. Section 4 examines more generally when ends are simple enough to guarantee uniqueness of end sums and 1-handle attaching. In dimensions 4 and up, it suffices for the end to satisfy the *Mittag-Leffler* condition (also called *semistability*), whose definition we recall in Section 4. Ends that are simply connected or topologically collared are Mittag-Leffler; in fact, the condition can only fail when the end requires infinitely many $(n-1)$-handles in any topological handle decomposition (Proposition 4.3). For example, Stein manifolds of complex dimension at least 2 have (unique) Mittag-Leffler ends.

The Mittag-Leffler condition is necessary and sufficient to guarantee that any two rays approaching the end are properly homotopic. This fact traces back at least to Geoghegan in the 1980s, and appears to have been folklore since the preceding decade (see Edwards and Hastings [EH76], for which Ross was consulted by Edwards, Mihalik [Mi83, Thm. 2.1], a student of Ross, and also Geoghegan [Ge08]). Unaware of Geoghegan's work and prompted by Siebenmann, Calcut and King worked out an algebraic classification of proper rays up to proper homotopy on an arbitrary end in 2002. This material was later excised from the 2012 published version of [CKS12] due to length considerations and since a similar proof had appeared in Geoghegan's text [Ge08] in the mean time.

The present paper gives a much simplified version of the proof, dealing only with the Mittag-Leffler case, in order to highlight the topology underlying the algebraic argument. We obtain a general statement (Theorem 4.5) about attaching countable collections of 1-handles to an open manifold. The following theorem is a special case.

Theorem 1.1. *Let X be a (possibly disconnected) n-manifold, $n \geq 4$. Then the result of attaching a (possibly infinite) collection of 1-handles at infinity to some oriented Mittag-Leffler ends of X depends only on the pairs of ends to which each 1-handle is attached, and whether the corresponding orientations agree.*

Note that uniqueness of end sums along Mittag-Leffler ends (preserving orientations) is a special case. Theorem 4.5 also deals with ends that are nonorientable or not Mittag-Leffler.

This theorem has consequences for open 4-manifold smoothing theory, which we explore in Section 5. The operation of end summing with an exotic $\mathbb{R}^4$ can be treated more systematically. The theorem shows that the monoid $\mathcal{R}$ acts on the set $\mathcal{S}(X)$ of smoothings of any 4-manifold X with a Mittag-Leffler end, and more generally a product of copies of $\mathcal{R}$ acts on $\mathcal{S}(X)$ through any countable collection of Mittag-Leffler ends (see Corollary 5.1). One can also deal with arbitrary ends by keeping track of a family of proper homotopy classes of rays. Similarly, one can act on $\mathcal{S}(X)$ by summing with exotic smoothings of $\mathbb{R}\times S^3$ along properly embedded lines (Corollary 5.4), or modify smoothings along properly embedded star-shaped graphs. While summing with a fixed exotic $\mathbb{R}^4$ is unique for an oriented (or nonorientable) Mittag-Leffler end, Section 3 suggests that there should be examples of nonuniqueness when the end of X is not Mittag-Leffler. However, such examples seem elusive, prompting the following natural question.

Question 1.2. *Let X be a smooth, oriented 4-manifold that is connected at infinity. Can end-summing X with a fixed exotic $\mathbb{R}^4$, preserving orientation, yield different diffeomorphism types depending on the choice of ray in X?*

We show (Proposition 5.3) that such examples would be quite difficult to detect.

Throughout the text, we take manifolds to be Hausdorff with countable basis, so with only countably many components. We allow boundary, and note that the theory is vacuous unless there is a noncompact component. *Open* manifolds are those with no boundary and no compact components. We work in a category CAT that can be DIFF, PL, or TOP. For example, DIFF homeomorphisms are the same as diffeomorphisms. Embeddings (particularly with codimension 0) are not assumed to be proper.

2. 1-handles at infinity

We begin with our procedure for attaching 1-handles at infinity. Recall that a CAT proper embedding $\gamma\colon Y^k \hookrightarrow X^n$ of CAT manifolds is called *flat* if it extends to a CAT embedding $\nu\colon Y^k \times \mathbb{R}^{n-k} \hookrightarrow X^n$. When all components of Y are contractible, this is automatically true in the smooth category, or in the PL category when $k = 1$. However, in the topological category, the hypothesis is necessary in order to avoid wild arcs for $n \geq 3$. Without loss of generality, we assume that ν extends to a proper embedding $Y \times D^{n-k} \hookrightarrow X$ for some smooth identification of $\mathbb{R}^{n-k}$ with the interior of the closed disk D^{n-k}. For example, such an embedding can be obtained from an arbitrary ν by passing to a disk bundle in the domain, with radii controlled using a proper function $X \to [0, \infty)$.

Definition 2.1. A *multiray* in a CAT n-manifold X is a flat CAT proper embedding $\gamma\colon S \times [0, \infty) \hookrightarrow X$ for some discrete (so necessarily countable) set S called the *index set* of γ. If the domain has a single component, γ will be called a *ray*. Given two multirays $\gamma^-, \gamma^+\colon S \times [0, \infty) \hookrightarrow X$ with disjoint images, choose disjoint extensions $\nu^\pm\colon S \times [0, \infty) \times \mathbb{R}^{n-1} \hookrightarrow X$ as above, and let Z be the CAT manifold obtained by gluing together X and $S \times [0, 1] \times \mathbb{R}^{n-1}$ by identifications $\nu^\pm \circ (\mathrm{id}_S \times \varphi^\pm \times \rho^\pm)$, where $\varphi^-\colon [0, \frac{1}{2}) \to [0, \infty)$ and $\varphi^+\colon (\frac{1}{2}, 1] \to [0, \infty)$ and $\rho^\pm\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are diffeomorphisms, with $\rho^\pm$ chosen so that $\varphi^\pm \times \rho^\pm$ preserves orientation. Then Z is obtained by *attaching 1-handles at infinity* to X along γ^- and γ^+ (see Figure 1).

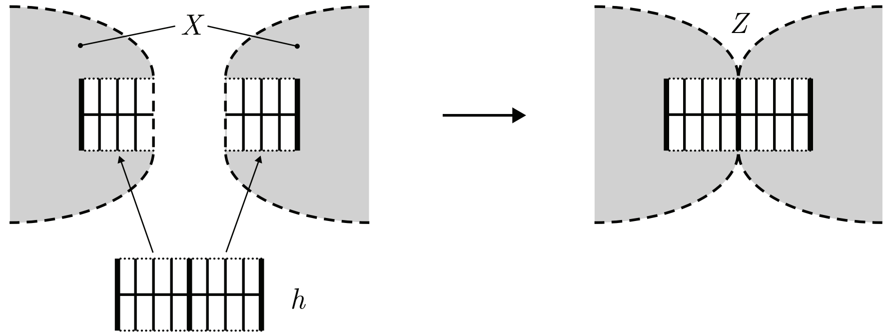

FIGURE 1. Data for addition of h, a 1-handle at infinity, to the n-manifold X (left) and resulting n-manifold Z (right).

We will see that Z depends on the choices of $\gamma^\pm$, but is independent of the choices of extensions $\nu^\pm$, diffeomorphisms $\varphi^\pm$ and $\rho^\pm$, and parametrization of the rays of $\gamma^\pm$. The domains of $\varphi^\pm$ can be replaced by smaller neighborhoods of the endpoints of $[0,1]$ without changing Z, making it more obvious that attaching compact 1-handles to the boundary of a compact manifold has the effect of attaching handles at infinity to the interior. Yet another description is to arrange the tubular neighborhoods of the multirays to have flat boundary, then remove the open neighborhoods and glue together the resulting $\mathbb{R}^{n-1}$ boundary components. The case of handle attaching where S is a single point and X has two components that are connected by the 1-handle at infinity is called the *end-sum* or *connected sum at infinity* in the literature.

Remark. Handles of higher index are also useful [G16], although additional subtleties arise. For example, a Casson handle can be attached to an unknot in the boundary of a 4-ball so that the interior of the resulting smooth 4-manifold is not diffeomorphic to the interior of any compact manifold. However, the interior of the Casson handle is diffeomorphic to $\mathbb{R}^4$, so we can interchange the roles of the two subsets, exhibiting the manifold as $\mathbb{R}^4$ with a 2-handle attached at infinity. The latter is attached along a properly embedded $S^1 \times [0,\infty)$ in $\mathbb{R}^4$ that is topologically unknotted but smoothly knotted, and cannot be smoothly compactified to an annulus in the closed 4-ball.

Variations on the above 1-handle construction are used in [G13]. Let X be a topological 4-manifold with a fixed smooth structure, and let R be an exotic $\mathbb{R}^4$ (a smooth manifold homeomorphic but not diffeomorphic to $\mathbb{R}^4$). Choose a smooth ray in X, and homeomorphically identify a smooth, closed tubular neighborhood N of it with the complement of a similar neighborhood in R. Transporting the smooth structure from R to N, where it fits together with the original one on $X - \operatorname{Int} N$, we obtain a new smooth structure on X diffeomorphic to an end-sum of X and R. The advantage of this description is that it fixes the underlying topological manifold, allowing us to assert, for example, that the two smooth structures are stably isotopic. Another variation from [G13] is to sum a smooth structure with an exotic $S^3 \times \mathbb{R}$ along a pair of smooth, properly embedded lines, one of which is topologically isotopic to $\{p\} \times \mathbb{R} \subset S^3 \times \mathbb{R}$. One can similarly change a smooth structure on a high-dimensional PL manifold by summing along a line with $\Sigma \times \mathbb{R}$ for some exotic sphere Σ. We exhibit these operations in Section 5 as well-defined monoid actions on the set of isotopy classes of smoothings of a fixed topological manifold. One can also consider CAT sums along lines in general. We discuss nonuniqueness of this latter operation in Section 3, to elucidate the corresponding discussion for 1-handles at infinity.

There are several obvious sources of nonuniqueness for attaching 1-handles at infinity. For attaching 1-handles in the compact setting, the result can depend both on orientations and on choices of boundary components. We will consider orientations in Section 4, but now consider the noncompact analogue of the set of boundary components, the space of ends of a manifold (e.g., [HR96]). Recall that for a manifold X (for example), a *neighborhood of infinity* is the complement of a compact set, and a *neighborhood system of infinity* is a nested sequence $\{U_i | i \in \mathbb{Z}^+\}$ of neighborhoods of infinity with empty intersection, and with the closure of U_{i+1} contained in U_i for all $i \in \mathbb{Z}^+$.

Definition 2.2. For a fixed neighborhood system $\{U_i\}$ of infinity, the *space of ends* of X is given by $\mathcal{E} = \mathcal{E}(X) = \lim_{\leftarrow} \pi_0(U_i)$.

That is, an end $\epsilon \in \mathcal{E}(X)$ is given by a sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$, where each V_i is a component of U_i. For two different neighborhood systems of infinity for X, the resulting spaces $\mathcal{E}(X)$ can be canonically identified: The set is preserved when we pass to a subsequence, but any two neighborhood systems of infinity have interleaved subsequences. A *neighborhood* of the end ϵ is an open subset of X containing one of the subsets V_i. This notion allows us to topologize the set $X \cup \mathcal{E}(X)$ so that X is homeomorphically embedded as a dense open subset and $\mathcal{E}(X)$ is totally disconnected [Fr31]. (The new basis elements are the components of each U_i, augmented by the ends of which they are neighborhoods.) The resulting space is Hausdorff with a countable basis. If X has only finitely many components, this space is compact, and called the *Freudenthal* or *end compactification* of X. In this case, $\mathcal{E}(X)$ is homeomorphic to a closed subset of a Cantor set.

Every ray γ in a manifold X determines an end $\epsilon_\gamma \in \mathcal{E}(X)$. This is because γ is proper, so every neighborhood U of infinity in X contains $\gamma([k,\infty))$ for sufficiently large k, and this image lies in a single component of U. In fact, an alternate definition of $\mathcal{E}(X)$ is as the set of equivalence classes of rays, where two rays are considered equivalent if their restrictions to $\mathbb{Z}^+$ are properly homotopic. A multiray $\gamma\colon\ S \times [0,\infty) \hookrightarrow X$ then determines a function $\epsilon_\gamma\colon\ S \to \mathcal{E}(X)$ that is preserved under proper homotopy of γ. Attaching 1-handles at infinity depends on these functions for γ^- and γ^+, just as attaching compact 1-handles depends on choices of boundary components, with examples of the former easily obtained from the latter by removing boundary. We will find more subtle dependence on the multirays in the next section, but a weak condition preventing these subtleties in Section 4.

3. Nonuniqueness

We now investigate examples of nonuniqueness in the simplest possible setting. In each case, we begin with a manifold X without boundary and with finitely many ends, and attach a single 1-handle at infinity, at a specified pair of ends. We assume the 1-handle respects a preassigned orientation on X. For attaching 1-handles in the compact setting, this would be enough information to uniquely specify the result, but we demonstrate that uniqueness can still fail for a 1-handle at infinity. It was shown in [CH14] that even the proper homotopy type need not be uniquely determined; Example 3.2 below sketches the simplest construction from that paper. Our subsequent examples are more subtle, having the same proper homotopy type but distinguished by their homeomorphism or diffeomorphism types. All of these examples necessarily have complicated fundamental group behavior at infinity, since Section 4 proves uniqueness when the fundamental group is suitably controlled. We obtain the required complexity by the following construction, which generalizes examples of [CH14]:

Definition 3.1. For an oriented CAT manifold X with two multirays $\gamma^-, \gamma^+\colon\ S \times [0,\infty) \hookrightarrow X$ whose images are disjoint, *ladder surgery* on X along γ^- and γ^+ is orientation-preserving surgery on the infinite family of 0-spheres given by $\{\gamma^-(s,n), \gamma^+(s,n)\}$ for each $s \in S$ and $n \in \mathbb{Z}^+$. That is, we find disjoint CAT balls with flat boundaries centered at the points $\gamma^\pm(s,n)$, remove the interiors of the balls, and glue each resulting pair of boundary spheres together by a reflection (so that the orientation of X extends).

It is not hard to verify that the resulting oriented diffeomorphism type only depends on the end functions $\epsilon_{\gamma^\pm}$ of the multirays; see Corollary 4.10 for details and a generalization to unoriented manifolds. If X has two components X_1 and X_2, each with k ends, any bijection from $\mathcal{E}(X_1)$ to $\mathcal{E}(X_2)$ determines a connected manifold with k ends obtained by ladder surgery with $S = \mathcal{E}(X_1)$. Such a manifold will be called a *ladder sum* of X_1 and X_2. For closed, connected, oriented $(n-1)$-manifolds M and N, we let $\mathbb{L}(M, N)$ denote the ladder sum of the two-ended manifolds $\mathbb{R} \times M$ and $\mathbb{R} \times N$, for the bijection preserving the ends of $\mathbb{R}$. (This is a slight departure from [CH14], which used the one-ended manifold $[0, \infty)$ in place of $\mathbb{R}$.) Note that a ladder surgery transforms the multirays $\gamma^\pm$ into infinite unions of circles, and surgery on all these circles (with any framings) results in the manifold obtained from X by adding 1-handles at infinity along $\gamma^\pm$. (This is easily seen by interpreting the surgeries as attaching 1- and 2-handles to $I \times X$.)

The examples in [CH14] are naturally presented in terms of ladder surgery and 1-handle addition at infinity. They also provide (in a natural sense) the simplest possible examples where a single 1-handle may be attached at infinity in essentially distinct ways, namely an orientation-preserving end-sum of one-ended manifolds.

Example 3.2 ([CH14])**.** For a fixed prime $p > 1$, let E denote the $\mathbb{R}^2$-bundle over S^2 with Euler number $-p$ (so E has a neighborhood of infinity diffeomorphic to $\mathbb{R} \times L(p, 1)$). Let Y be the ladder sum of E and $\mathbb{R}^4$ using rays γ^- in E and γ^+ in $\mathbb{R}^4$. We will attach a single 1-handle at infinity to the disjoint union $X = Y \sqcup E$ in two ways to produce distinct, one-ended, boundaryless manifolds Z_0 and Z_1. Let γ_0 be a ray in Y lying in E and parallel to γ^-. Let γ_1 be a ray in Y lying in the $\mathbb{R}^4$ summand and parallel to γ^+. Let γ be any ray in E, and let Z_i be obtained from X by attaching a 1-handle at infinity along γ_i and γ. The manifolds Z_0 and Z_1 are not properly homotopy equivalent (in fact, their ends are not properly homotopy equivalent) since they have nonisomorphic cohomology algebras at infinity [CH14]. The basic idea is that both manifolds Z_i have obvious splittings as ladder sums. For Z_0, one summand is $\mathbb{R}^4$, so all cup products from $H^1(Z_0; \mathbb{Z}/p) \otimes H^2(Z_0; \mathbb{Z}/p)$ are supported in the other summand in a 1-dimensional subspace of $H^3(Z_0; \mathbb{Z}/p)$. However, Z_1 has cup products on both sides, spanning a 2-dimensional subspace.

Our remaining examples are pairs with the same homotopy type, distinguished by more subtle means.

Example 3.3. We now show that end-summing along a fixed pair of ends can produce properly homotopy equivalent manifolds with different homeomorphism types, and that summing along a line can produce similar results. Let P and Q, respectively, denote $\mathbb{C}P^2$ and Freedman's fake $\mathbb{C}P^2$ (e.g. [FQ90]). Then there is a homotopy equivalence between P and Q, restricting to a pairwise homotopy equivalence between the complements of a ball interior in each. But P and Q cannot be homeomorphic since Q is unsmoothable. The ladder sum $\mathbb{L}(P,Q)$ is an unsmoothable topological 5-manifold with two ends. The lines $\mathbb{R}\times\{p\}\subset\mathbb{R}\times P$ and $\mathbb{R}\times\{q\}\subset\mathbb{R}\times Q$ can be chosen to lie in $\mathbb{L}(P,Q)$, with each spanning the two ends of $\mathbb{L}(P,Q)$, but they are dual to two different elements of $H^4(\mathbb{L}(P,Q);\mathbb{Z}/2)$ (cf. [CH14]), with $\mathbb{R}\times\{q\}$ dual to the Kirby-Siebenmann smoothing obstruction of $\mathbb{L}(P,Q)$. Clearly, there is a proper homotopy equivalence of $\mathbb{L}(P,Q)$ interchanging the two lines. Thus, the two resulting ways to sum $\mathbb{L}(P,Q)$ along a line with $\mathbb{R}\times\overline{Q}$ (where the orientation on Q is reversed for later convenience) give properly homotopy equivalent manifolds, namely $\mathbb{L}(\overline{Q}\#P,Q)$ and $L(P,Q\#\overline{Q})=L(P,P\#\overline{P})$. (The last equality follows from Freedman's classification of simply connected topological 4-manifolds [FQ90].) These two manifolds cannot be homeomorphic, since the latter is a smooth manifold whereas the former is unsmoothable, with Kirby-Siebenmann obstruction dual to a pair of lines running along opposite sides of the ladder. (A discussion of the cohomology of such manifolds can be found in [CH14], but more simply, there are obvious embedded copies of Q on which the Kirby-Siebenmann obstruction evaluates nontrivially.)

While it is not surprising that summing on lines representing different cohomology classes can give nonhomeomorphic manifolds, this example can also be used to elucidate end-sums. Instead of summing along a line, we can end-sum $\mathbb{L}(P,Q)$ with $\mathbb{R}\times\overline{Q}$ along their positive ends in two different ways (using rays obtained from the positive ends of the previous lines). We obtain a pair of properly homotopy equivalent, unsmoothable, three-ended manifolds. In one case, the modified end has a neighborhood that is smoothable, and in the other case, all three ends fail to have smoothable neighborhoods since the Kirby-Siebenmann obstruction cannot be avoided. Thus, we have a pair of nonhomeomorphic, but properly homotopy equivalent, manifolds, both obtained by an orientation-preserving end sum on the same pair of ends.

There are several other variations of the construction. We can replace the $\mathbb{R}$ factor by $[0,\infty)$, so that the ladder sum is one-ended, to get an example of nonuniqueness of summing one-ended topological manifolds with compact boundary. Unfortunately, we cannot cap off the boundaries to obtain one-ended open manifolds, since the Kirby-Siebenmann obstruction is a cobordism invariant of topological 4-manifolds. However, we can modify the original ladder sum so that we do ladder surgery on the positive end, but end sum on the negative end (which then has a neighborhood homeomorphic to $\mathbb{R} \times (\overline{P} \# \overline{Q})$). Now we have a connected, two-ended open manifold such that the ends can be joined by an orientation-preserving 1-handle at infinity in two different ways, yielding properly homotopy equivalent but nonhomeomorphic one-ended manifolds.

In higher dimensions, the Kirby-Siebenmann obstruction of a neighborhood V of an end cannot be killed by adding 1-handles at infinity (since $H^4(V; \mathbb{Z}/2)$ is not disturbed), but we can do the analogous construction using higher smoothing obstructions. This time, we obtain PL n-manifolds (for some $n \geq 9$) that are properly homotopy equivalent but not PL homeomorphic. Let P and Q be homotopy equivalent PL $(n-1)$-manifolds with P and $Q - \{q_0\}$ smooth but Q unsmoothable. (For an explicit 24-dimensional pair, see Proposition 5.1 of [A68].) The previous discussion applies almost verbatim with PL in place of TOP, with the smoothing obstruction in $H^{n-1}(X; \Theta_{n-2})$ for PL manifolds X in place of the Kirby-Siebenmann obstruction. The one change is that smoothability of $Q \# \overline{Q}$ follows since it is the double of the smooth manifold obtained from Q by removing the interior of a PL ball centered at q_0. (This time the orientation reversal is necessary since the smoothing obstruction need not have order 2.)

Example 3.4. A similar construction shows that end-summing along a fixed pair of ends can produce PL homeomorphic but nondiffeomorphic manifolds. Let Σ be an exotic $(n-1)$-sphere with $n > 5$. The ladder sum $\mathbb{L}(\Sigma, S^{n-1})$ is then a two-ended smooth manifold with a PL self-homeomorphism that is not isotopic to a diffeomorphism. Since $\Sigma \# \overline{\Sigma} = S^{n-1}$, summing $\mathbb{L}(\Sigma, S^{n-1})$ along a line with $\mathbb{R} \times \overline{\Sigma}$ gives the two manifolds $\mathbb{L}(S^{n-1}, S^{n-1})$ and $\mathbb{L}(\Sigma, \overline{\Sigma})$. The first of these bounds an infinite handlebody made with 0- and 1-handles, as does its universal cover. Since a contractible 1-handlebody is a ball with some boundary points removed, it follows that the universal cover of $\mathbb{L}(S^{n-1}, S^{n-1})$ embeds in S^n. However, $\mathbb{L}(\Sigma, \overline{\Sigma})$ contains copies of Σ arbitrarily close to its ends. Since any homotopy $(n-1)$-sphere $(n > 5)$ that embeds in S^n cuts out a ball, so is standard, it follows that no neighborhood of either end of $\mathbb{L}(\Sigma, \overline{\Sigma})$ has universal cover embedding in S^n. Thus, the two manifolds have nondiffeomorphic ends, although they are PL-homeomorphic. As before, we can modify this example to get a pair of end-sums of two-ended manifolds, or a pair obtained from a two-ended connected manifold by joining the ends with a 1-handle in two different ways. This time however, we can also interpret the example as end-summing two one-ended open manifolds, by first obtaining one-ended manifolds with compact boundary, then capping off the boundary. (Note that Σ bounds a compact manifold. Unlike codimension-0 smoothing existence obstructions, the uniqueness obstructions are not cobordism invariants.)

This construction has an analogue in dimension 4, where the categories DIFF and PL coincide. Replace $\mathbb{R} \times \Sigma$ by W, Freedman's exotic $\mathbb{R} \times S^3$. This is distinguished from the standard $\mathbb{R} \times S^3$ by the classical PL uniqueness obstruction in $H^3(\mathbb{R} \times S^3; \mathbb{Z}/2) \cong \mathbb{Z}/2$, dual to $\mathbb{R} \times \{p\}$. The ladder sum L of W with $\mathbb{R} \times S^3$ can be summed along a line with W in two obvious ways. These can be interpreted as smoothings on the underlying topological manifold $\mathbb{L}(S^3, S^3)$. The smoothings are non-isotopic (even stably, i.e., after Cartesian product with $\mathbb{R}$), since the uniqueness obstruction by which they differ is dual to a pair of lines on opposite sides of the ladder. However, the authors have not been able to distinguish their diffeomorphism types. The problem with the previous argument is that the sum of two copies of W along a line is not diffeomorphic to $\mathbb{R} \times S^3$ (although the classical invariant vanishes). While W contains a copy of the Poincaré homology sphere Σ separating its ends, so cannot embed in S^4, the sum of two copies of W contains $\Sigma \# \Sigma$, which also does not embed in S^4. The effect of summing with reversed orientation or switched ends, or replacing Σ by a different homology sphere, is less clear. This leads to the following question, which is discussed further in Section 4 (Question 5.5).

Question 3.5. *Are there two exotic smoothings on $\mathbb{R} \times S^3$ whose sum along a line is the standard $\mathbb{R} \times S^3$?*

If such smoothings exist, one of which has the additional property that every neighborhood of one end has a slice $(a, b) \times S^3$ that cannot smoothly embed in S^4, then there are two one-ended open 4-manifolds that can be end-summed in two homeomorphic but not diffeomorphic (or PL homeomorphic) ways.

4. Uniqueness for Mittag-Leffler ends

Having examined the failure of uniqueness in the last section, we now look for hypotheses that guarantee that 1-handle attaching at infinity is unique. There are several separate issues to deal with. In the compact setting, attaching a 1-handle to given boundary components can yield two different results if both boundary components are orientable, so uniqueness requires specified orientations in that case. The same issue arises for 1-handles at infinity. Beyond that, we must consider the dependence on the involved multirays. Since rays in $\mathbb{R}^3$ can be knotted, uncountably many homeomorphism types of contractible manifolds arise as end-sums of two copies of $\mathbb{R}^3$ [My99]. (See also [CH14].) Thus, we assume more than 3 dimensions and conclude, not surprisingly, that the multirays affect the result only through their proper homotopy classes, and that the choices of extensions to tubular neighborhoods cause no additional difficulties. We have already seen that different rays determining the same end can still lead to different results for end-summing with another fixed manifold and ray, but we find a weak group-theoretic condition on an end that entirely eliminates dependence on the choice of rays limiting to that end.

We begin with terminology for orientations. We will call an end ϵ of a manifold X *orientable* if it has an orientable neighborhood in X. An orientation on one connected, orientable neighborhood of ϵ determines an orientation on every other such neighborhood, through the component of their intersection that is a neighborhood of ϵ. Such a compatible choice of orientations will be called an *orientation of* ϵ, so every orientable end has two orientations. We let $\mathcal{E}_{\mathrm{O}} \subset \mathcal{E}(X)$ denote the open subset of orientable ends of X. (This need not be closed, as seen by deleting a sequence of points of X converging to a nonorientable end.) For a map $\gamma\colon S \times [0,\infty) \to X$ (S discrete), if X is smooth its tangent bundle pulls back to a trivial bundle γ^*TX over $S \times [0,\infty)$. An orientation on this bundle will be called a *local orientation of* X *along* γ, and if such an orientation is specified, γ will be called *locally orienting*. If X is PL or topological, we use the same terminology, using the appropriate analogue of the tangent bundle, or equivalently but more simply, using local homology $H_n(X, X - \{\gamma(s,t)\}) \cong \mathbb{Z}$. A homotopy γ_t of a locally orienting map canonically extends to a 1-parameter family of locally orienting maps, allowing us to compare local orientations on γ_0 and γ_1. If a ray γ determines an orientable end $\epsilon_\gamma \in \mathcal{E}_{\mathrm{O}}$, then a local orientation along γ induces an orientation on the end, since $\gamma([k,\infty))$ lies in a connected, oriented neighborhood of ϵ_γ when k is sufficiently large.

We now turn to the group theory of ends. See [Ge08] for a more detailed treatment. An *inverse sequence of groups* is a sequence $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$ of groups and homomorphisms. We suppress the homomorphisms from the notation, since they will be induced by obvious inclusions in our applications. A *subsequence* of an inverse sequence is another inverse sequence obtained by passing to a subsequence of the groups and using the obvious composites of homomorphisms. Passing to a subsequence, and its inverse procedure, together generate the standard notion of equivalence of inverse sequences.

Definition 4.1. An inverse sequence $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$ of groups is called *Mittag-Leffler* (or *semistable*) if for each $i \in \mathbb{Z}^+$ there is a $j \geq i$ such that all G_k with $k \geq j$ have the same image in G_i.

Clearly, a subsequence is Mittag-Leffler if and only if the original sequence is, so the notion is preserved by equivalences. After passing to a subsequence, we may assume $j = i + 1$ in the definition.

For a CAT manifold X with a ray γ and a neighborhood system $\{U_i\}$ of infinity, we can always reparametrize γ so that $\gamma([i, \infty))$ lies in U_i for each $i \in \mathbb{Z}^+$. (It is more traditional to use proper maps here, but we lose nothing by restricting to embedded rays.) Then we have:

Definition 4.2. The *fundamental progroup* of X based at γ is the inverse sequence of groups $\pi_1(U_i, \gamma(i))$, where the homomorphism $\pi_1(U_{i+1}, \gamma(i+1)) \to \pi_1(U_i, \gamma(i))$ is the inclusion-induced map to $\pi_1(U_i, \gamma(i + 1))$ followed by the isomorphism moving the base point to $\gamma(i)$ along the arc $\gamma|[i, i + 1]$.

Passing to a subsequence of $\{U_i\}$ replaces the fundamental progroup by a subsequence of it. Since any two neighborhood systems of infinity have interleaved subsequences, the fundamental progroup is independent, up to equivalence, of the choice of neighborhood system. It is routine to check that it is similarly preserved by any proper homotopy of γ, so it only depends on X and the proper homotopy class of γ. Furthermore, the inverse sequence is unchanged if we replace each U_i by its connected component containing $\gamma([i, \infty))$, so it is equivalent to use a neighborhood system of the end ϵ_γ. Beware, however, that even if there is only one end, the choice of γ can affect the fundamental progroup, and even whether its inverse limit vanishes. (See [Ge08] Example 16.2.4. The homomorphisms in the example are injective, but changing γ conjugates the resulting nested subgroups, changing their intersection.)

We call the pair (X, γ) *Mittag-Leffler* if its fundamental progroup is Mittag-Leffler. We will see in Lemma 4.8(a) below that this condition implies γ is determined up to proper homotopy by its induced end ϵ_γ, so the fundamental progroup of ϵ_γ is well-defined in this case, and it makes sense to call an end of a manifold Mittag-Leffler. Note that this condition rules out ends made by ladder surgery, and hence the examples of Section 3. We will denote the set of Mittag-Leffler ends of X by $\mathcal{E}_{\mathrm{ML}} \subset \mathcal{E}(X)$, and its complement by $\mathcal{E}_{\mathrm{bad}}$.

Many important types of ends are Mittag-Leffler. *Simply connected* ends are (essentially by definition) the special case for which the given images all vanish. *Topologically collared* ends, with a neighborhood homeomorphic to $\mathbb{R} \times M$ for some compact $(n-1)$-manifold M, are *stable*, the special case for which the fundamental progroup is equivalent to an inverse sequence with all maps isomorphisms. In the smooth category, we can analyze ends using a Morse function φ that is exhausting (i.e., proper and bounded below). For such a function, the preimages $\varphi^{-1}(i, \infty)$ for $i \in \mathbb{Z}^+$ form a neighborhood system of infinity.

Proposition 4.3. *Let X be a smooth n-manifold. If an end ϵ of X is not Mittag-Leffler, then for every exhausting Morse function φ on X and every $t \in \mathbb{R}$, there are infinitely many critical points of index $n-1$ in the component of $\varphi^{-1}(t, \infty)$ containing ϵ. In particular, if X admits an exhausting Morse function with only finitely many index-$(n-1)$ critical points, then all of its ends are Mittag-Leffler.*

Proof. After perturbing φ and composing it with an orientation-preserving diffeomorphism of $\mathbb{R}$, we can assume each $\varphi^{-1}[i, i+1]$ is an elementary cobordism. Let $\bar{V}_i$ be the component of $\varphi^{-1}[i, \infty)$ containing ϵ. Since ϵ is not Mittag-Leffler, the corresponding fundamental progroup must have infinitely many homomorphisms that are not surjective. Thus, there are infinitely many values of i for which $\bar{V}_i$ is made from $\bar{V}_{i+1}$ by attaching a 1-handle. Equivalently, the cobordism $\bar{V}_i - \operatorname{Int} \bar{V}_{i+1}$ is built from its boundary components in $\varphi^{-1}(i)$ with an $(n-1)$-handle, so it contains an index $n-1$ critical point of φ. □

Since every Stein manifold of complex dimension m (real dimension $2m$) has an exhausting Morse function with indices at most m, we have:

Corollary 4.4. *For every Stein manifold whose complex dimension is not 1, the unique end of each component is Mittag-Leffler.* □

Since the Mittag-Leffler condition on an end of a CAT manifold is determined by the underlying topological manifold (in fact, by its proper homotopy type), we are free to change the smooth structure on a manifold before looking for a suitable Morse function. This is especially useful in dimension 4. For example, an exhausting Morse function on an exotic $\mathbb{R}^4$ with nonzero Taylor invariant must have infinitely many index-3 critical points [T97], but after passing to the standard structure, there is a Morse function with a unique critical point. (Furthermore, an exotic $\mathbb{R}^4$ is topologically collared and simply connected at infinity.) Proposition 4.3 is most generally stated using topological Morse functions on topological manifolds. (These are well-behaved

[KS77] and can be constructed from handle decompositions, which exist on all open topological manifolds, e.g. [FQ90].)

The main point of our uniqueness theorem is that when we attach 1-handles at infinity, any locally orienting defining ray that determines a Mittag-Leffler end will affect the outcome only through the end and local orientation it determines. If the end is also nonorientable, then even the local orientation has no influence. To state this in full generality, we also allow rays determining ends that are not Mittag-Leffler, which are required to remain in a fixed proper homotopy class. That is, we allow an arbitrary multiray γ, but require its restriction to the subset $\epsilon_\gamma^{-1}(\mathcal{E}_{\mathrm{bad}})$ of the index set S (corresponding to rays determining ends that are not Mittag-Leffler) to lie in a fixed proper homotopy class. For 1-handles with at least one defining ray determining a nonorientable Mittag-Leffler end, no further constraint is necessary, but otherwise we keep track of orientations. For rays we have constrained to a single proper homotopy class, the local orientation does this. For orientable Mittag-Leffler ends, we are not given a proper homotopy, but can work relative to an orientation of the end. We obtain:

Theorem 4.5. *For a* CAT *n-manifold X with $n \geq 4$, discrete S and $i = 0, 1$, let $\gamma_i^-, \gamma_i^+ \colon S \times [0,\infty) \hookrightarrow X$ be locally orienting* CAT *multirays with disjoint images such that the end functions $\epsilon_{\gamma_i^\pm} \colon S \to \mathcal{E}(X)$ are independent of i. Suppose that*

(a) *the restrictions of γ_0^- and γ_1^- to the index subset $\epsilon_{\gamma_0^-}^{-1}(\mathcal{E}_{\mathrm{bad}})$ are properly homotopic,*

(b) *for each $s \in \epsilon_{\gamma_0^-}^{-1}(\mathcal{E}_{\mathrm{bad}} \cup \mathcal{E}_{\mathrm{O}}) \cap \epsilon_{\gamma_0^+}^{-1}(\mathcal{E}_{\mathrm{bad}} \cup \mathcal{E}_{\mathrm{O}})$, the local orientations of the corresponding rays in γ_0^- and γ_1^- induce the same orientation of the end if there is one, and otherwise correspond under the proper homotopy given in (a).*

(c) *the two analogous conditions apply to γ_i^+.*

Let Z_i be the result of attaching 1-handles to X along $\gamma_i^\pm$ (for any choice of extension $\nu_i^\pm$). Then there is a CAT *homeomorphism from Z_0 to Z_1 sending the submanifold X onto itself by a* CAT *homeomorphism* CAT *ambiently isotopic in X to the identity map.*

It follows that 1-handle attaching is not affected by reparametrization of the rays (a proper homotopy), or changing the auxiliary diffeomorphisms $\varphi^\pm$ and $\rho^\pm$ occurring in the definition (which only results in changing the parametrization and extension, respectively).

Corollary 4.6. *For an oriented* CAT *n-manifold X with $n \geq 4$, every countable family of pairs of Mittag-Leffler ends canonically determines* *a* CAT *manifold obtained from X by attaching 1-handles at infinity to those pairs of ends, respecting the orientation.* □

Corollary 4.7. *Every manifold obtained from a Stein manifold X by attaching 1-handles at infinity, respecting the complex orientation, admits a Stein structure.*

Proof. Since every open, oriented surface admits a Stein structure, we assume X has real dimension $2m \geq 4$. Since X is Stein, it has an exhausting Morse function with indices at most m. It can then be described as the interior of a smooth handlebody whose handles have index at most m. This is well-known when there are only finitely many critical points. A proof of the infinite case is given in the appendix of [G09], which also shows that when $m = 2$ one can preserve the extra framing condition that arises for 2-handles. By Corollaries 4.4 and 4.6, we can realize the 1-handles at infinity by attaching compact handles to the handlebody before passing to the interior (and after adding infinitely many canceling 0-1 pairs if necessary to accommodate infinitely many new 1-handles, avoiding compactness issues). Now we can convert the handlebody interior back into a Stein manifold by Eliashberg's Theorem (see [CE12]). □

The proof of Theorem 4.5 follows from two lemmas. First we prove that multirays with a given Mittag-Leffler end function are unique up to proper homotopy.

Lemma 4.8 (a)**.** *If (X, γ) is a* CAT *Mittag-Leffler pair, then every ray determining the same end as γ is properly homotopic to γ. In particular, the Mittag-Leffler condition for ends is independent of choice of ray, so the subset $\mathcal{E}_{\mathrm{ML}} \subset \mathcal{E}$ is well-defined.*

(b) *Let $\gamma_0, \gamma_1\colon S \times [0,\infty) \hookrightarrow X$ be locally orienting multirays in a* CAT *manifold, with the same end function. Suppose that this function $\epsilon_{\gamma_0} = \epsilon_{\gamma_1}$ has image in $\mathcal{E}_{\mathrm{ML}}$, and that for each s with $\epsilon_{\gamma_0}(s) \in \mathcal{E}_{\mathrm{O}}$, the corresponding locally orienting rays of γ_0 and γ_1 induce the same orientation (depending on s) of the end $\epsilon_{\gamma_0}(s)$. Then there is a proper homotopy from γ_0 to γ_1, respecting the given local orientations.*

The first sentence and its converse are essentially Proposition 16.1.2 of [Ge08], which is presented as an immediate consequence of two earlier statements: Proposition 16.1.1 asserts that the set of proper homotopy classes of rays approaching an arbitrary end corresponds bijectively to the derived limit $\lim^1_{\leftarrow} \pi_1(U_i, \gamma(i))$ of a neighborhood system U_i of infinity; Theorem 11.3.2 asserts that an inverse sequence of countable groups G_i is Mittag-Leffler if and only if $\lim^1_{\leftarrow} G_i$ has only one element.

We follow those proofs but considerably simplify the argument, eliminating use of derived limits, by focusing on the Mittag-Leffler case. This reveals the underlying geometric intuition: For a product end ϵ, i.e., when there is a neighborhood of ϵ homeomorphic to $\mathbb{R} \times M$, use the proper homotopy that is the identity on the first factor, and on the second precomposes with the homotopy $f_s(t) = (1-s)t$ of $[0,\infty)$. Properness is preserved through the first factor, and any problems are pushed out to infinity. If ϵ only has a neighborhood system with π_1-surjective inclusions, first properly homotope the rays to agree on $\mathbb{Z}^+ \subset [0,\infty)$, so they agree up to a proper sequence of loops. Surjectivity again allows us to push these out to infinity. In the general Mittag-Leffler case, we still have enough surjectivity to push each loop to infinity after pulling it back a single level in the neighborhood system (with properness preserved because we only pull back one level). The following proof efficiently encodes this procedure with algebra.

Proof. First we prove (a), showing that an arbitrary ray γ' determining the same Mittag-Leffler end as γ is properly homotopic to it. We also keep track of preassigned local orientations along the two rays. If ϵ_γ is orientable, we assume these local orientations induce the same orientation on ϵ_γ (as in (b)). Let $\{U_i\}$ be a neighborhood system of infinity, arranged (by passing to a subsequence if necessary) so that each j is $i+1$ in the definition of the Mittag-Leffler condition, and that the component of U_1 containing ϵ_γ is orientable if ϵ_γ is. Then reparametrize γ so that each $\gamma([i,\infty))$ lies in U_i. Reparametrize γ' similarly, then arrange it to agree with γ on $\mathbb{Z}^+$ by inductively moving γ' near each $i \in \mathbb{Z}^+$ separately, with compact support inside U_i. The limiting homotopy is then well-defined and proper. If ϵ_γ is nonorientable, then so is the relevant component of each U_i, so we can assume (changing the homotopy via orientation-reversing loops as necessary) that the local orientations along the two rays agree at each i. (This is automatic when ϵ_γ is orientable.) The two rays now differ by a sequence of orientation-preserving loops, representing classes $x_i \in \pi_1(U_i, \gamma(i))$ for each $i \geq 1$. Inductively choose orientation-preserving classes $y_i \in \pi_1(U_i, \gamma(i))$ for all $i \geq 2$ starting from an arbitrary y_2, and for $i \geq 1$ choosing $y_{i+2} \in \pi_1(U_{i+2}, \gamma(i+2))$ to have the same image in $\pi_1(U_i, \gamma(i))$ as $x_{i+1}^{-1} y_{i+1} \in \pi_1(U_{i+1}, \gamma(i+1))$. (This is where the Mittag-Leffler condition is necessary.) For each $i \geq 1$, let $z_i = x_i y_{i+1} \in \pi_1(U_i, \gamma(i))$ (where we suppress the inclusion map). In that same group, we then have $z_i z_{i+1}^{-1} = x_i y_{i+1} y_{i+2}^{-1} x_{i+1}^{-1} = x_i$. After another proper homotopy, we can assume the two rays and their induced local orientations on X agree along $\frac{1}{2}\mathbb{Z}^+$ and give the sequence

$z_1, z_2^{-1}, z_2, z_3^{-1}, \ldots$ in $U_1, U_1, U_2, U_2, \ldots$. Now a proper homotopy fixing $\mathbb{Z}^+ + \frac{1}{2}$ cancels all loops between these points, so that the two rays coincide. This completes the proof of (a), and also (since $\mathcal{E}_{\mathrm{ML}}$ is now well-defined) (b) when S is a single point.

For the general case of (b), fix a neighborhood system $\{U_i\}$ as before. We wish to apply the previous argument to each pair of of rays separately, letting γ be the given ray in γ_0. (Note that each resulting (X, γ) is now a Mittag-Leffler pair.) After reparametrizing each ray suitably, we can assume $[1, \infty)$ maps into an orientable component of some U_i whenever the associated end is orientable. Since the multirays are proper, each U_i contains all but finitely many pairs of rays. Thus, for each pair, we can apply the previous paragraph with the indexing of $\{U_i\}$ shifted, so that the argument applies to all pairs simultaneously with each step of each inductive argument involving only finitely many rays. This guarantees that properness is maintained, so the proof works as before. □

Remark. To see the correspondence with the geometric description, first consider the case with all inclusion maps π_1-surjective. Then the argument simplifies: We can just define $z_1 = 1$, and inductively choose z_{i+1} to be any pullback of $x_i^{-1} z_i$. Then z_i is a pullback of $(x_1 \cdots x_{i-1})^{-1}$, exhibiting the loops being transferred toward infinity.

To upgrade our proper homotopy to an ambient isotopy, we need the following lemma.

Lemma 4.9. *Suppose that X is a* CAT *n-manifold with $n \geq 4$ and Y is a* CAT *1-manifold with a closed subset $C \subset \operatorname{Int} Y$. Let $\Gamma\colon I \times Y \hookrightarrow \operatorname{Int} X$ be a topological proper homotopy, between* CAT *embeddings γ_0 and γ_1 that extend to* CAT *embeddings $\nu_i\colon Y \times \mathbb{R}^{n-1} \hookrightarrow X$ whose local orientations correspond under Γ. Then there is a* CAT *ambient isotopy $\Phi\colon I \times X \to X$, supported in a preassigned neighborhood of $\operatorname{Im} \Gamma$, such that $\Phi_0 = \mathrm{id}_X$ and $\Phi_1 \circ \nu_0$ agrees with ν_1 on a neighborhood of $C \times \{0\}$ in $Y \times \mathbb{R}^{n-1}$.*

To see the subtlety of this lemma, note that the corresponding statement in $\mathbb{R}^3$ is false even with Γ a proper isotopy of $Y = \mathbb{R}$: The isotopy Γ can slide a knot out to infinity, changing the fundamental group of the complement, and this can even be done while fixing the integer points of $\mathbb{R}$.

Proof. We modify Γ in stages. To avoid technicalities regarding flatness of $\nu_i(\partial Y \times \mathbb{R}^{n-1})$, we shorten Y slightly, removing a neighborhood of ∂Y disjoint from C. Then we apply CAT general position: After an ambient isotopy of γ_0 (which becomes the first stage of Φ), we can assume the images of γ_0 and γ_1 are disjoint. After a further C^0-small (hence proper) isotopy of Γ rel $\{0,1\}\times Y$, we can assume Γ is a flat CAT embedding of $I\times Y$, except for isolated double points when $n=4$. Since the image of Γ is now a 2-complex, it has a neighborhood W with no smoothing obstructions, and we can upgrade the CAT structure on W to a smooth structure into which Γ is a smooth map of some smoothing of Y. (Use the embeddings ν_i to define the smoothing near $\Gamma|(\{0,1\}\times Y)$, extend the smoothing over W, then smooth Γ by a C^0-small homotopy rel $\{0,1\}\times Y$. This will introduce more double points when $n=4$.) Now decompose Y as a cell complex with 0-skeleton Y_0. We can assume Γ restricts to a smooth embedding on some neighborhood N of $I\times Y_0$. Define Φ on $[0,\frac{1}{2}]\times Y$ by extending $\Gamma|N$ to a smooth ambient isotopy supported in W, working in disjoint compact neighborhoods of the components of N so that the Isotopy Extension Theorem applies. Now it suffices to assume Γ fixes a neighborhood of Y_0, and view Γ as a countable collection of path homotopies of the 1-cells of Y. We need the resulting immersed 2-disks to be disjoint. This is automatic when $n>4$, but is the step that fails for knotted lines in $\mathbb{R}^3$. For $n=4$, we push the disks off of each other by finger moves. This operation preserves properness of Γ since each compact subset of X intersects only finitely many disks, which have only finitely many intersections with other disks (and we do not allow finger moves over other fingers). Now we can extend to an ambient isotopy, working in disjoint compact neighborhoods of the disks. Since the construction is smooth in W, the proof is completed via uniqueness of tubular neighborhoods. □

Proof of Theorem 4.5. For each $i=0,1$, the two multirays γ_i^- and γ_i^+ can be thought of as a single multiray γ_i with index set $S^*=S\times\{-1,1\}$. For each index $(s,\sigma)\in\epsilon_{\gamma_0}^{-1}(\mathcal{E}_{\mathrm{O}})\subset S^*$, we arrange for the corresponding locally orienting rays in γ_0 and γ_1 to induce the same orientation of the end: If this is not already true, then Hypothesis (b) of the theorem implies that the opposite end $\epsilon_{\gamma_0}(s,-\sigma)$ is Mittag-Leffler but nonorientable. In this case, reverse the local orientations along both rays in γ_1 parametrized by s. This corrects the orientations without changing Z_1, since the change extends as a reflection of the 1-handle $\{s\}\times[0,1]\times\mathbb{R}^{n-1}$. Now split γ_i into two multirays γ_i^{ML} and γ_i^{bad}, according to whether the rays determine Mittag-Leffler ends. By Hypothesis (a), we have a proper homotopy from γ_0^{bad} to γ_1^{bad}, which respects the local orientations by Hypothesis (b) after possible flips when the opposite end is Mittag-Leffler but nonorientable. Lemma 4.8 then gives a proper homotopy from γ_0^{ML} to γ_1^{ML} respecting local orientations.

Reassembling the multirays, we obtain a proper homotopy from γ_0 to γ_1 that respects local orientations. Now we apply Lemma 4.9 with $Y = S^* \times [0,\infty)$, $C = S^* \times [1,\infty)$, and ν_i being the given extension of γ_i. We obtain a CAT ambient isotopy Φ of id_X such that $\Phi_1 \circ \nu_0$ agrees with ν_1 on a neighborhood N of $S^* \times [1,\infty)$. Note that the quotient space Z_i does not change if we cut back the 1-handles $S^* \times [0,1] \times \mathbb{R}^{n-1}$ to any neighborhood N' of $S^* \times \{\frac{1}{2}\} \times \mathbb{R}^{n-1}$ and use the restricted gluing map. Recall that the gluing map factors through an $\mathbb{R}^{n-1}$-bundle map $\mathrm{id}_{S^*} \times \varphi^\pm \times \rho^\pm$ to $S^* \times [0,\infty) \times \mathbb{R}^{n-1}$. We can assume that the resulting image of N' lies in some disk bundle inside $S^* \times [1,\infty) \times \mathbb{R}^{n-1}$. A smooth ambient isotopy supported inside a larger disk bundle moves this image into N. Conjugating with ν_i gives a CAT ambient isotopy $\Psi_{(i)}$ on X. Then $\Phi' = \Psi_{(1)}^{-1} \circ \Phi \circ \Psi_{(0)}$ is a CAT ambient isotopy for which $\Phi'_1 \circ \nu_0$ agrees with ν_1 on N'. The CAT homeomorphism Φ'_1 extends to one sending Z_0 to Z_1 with the required properties. □

We can now address uniqueness of ladder surgeries. Note that their definition immediately extends to unoriented manifolds, provided that we use locally orienting multirays.

Corollary 4.10. *For a* CAT *manifold X, discrete S and $i = 0, 1$, let $\gamma_i^\pm \colon S \times [0,\infty) \hookrightarrow X$ be locally orienting* CAT *multirays with disjoint images such that the end functions $\epsilon_{\gamma_i^\pm} \colon S \to \mathcal{E}(X)$ are independent of i. Suppose that for each $s \in \epsilon_{\gamma_0^-}^{-1}(\mathcal{E}_{\mathrm{O}}) \cap \epsilon_{\gamma_0^+}^{-1}(\mathcal{E}_{\mathrm{O}})$, the local orientations of the corresponding rays in $\gamma_i^\pm$ induce the same orientation of the end for $i = 0, 1$. Then the manifolds Z_i obtained by ladder surgery on X along $\gamma_i^\pm$ are* CAT *homeomorphic.*

Proof. We can assume that each ray of $\gamma_0^\pm$ determining an orientable end induces the same orientation of that end as the corresponding ray of $\gamma_1^\pm$, after reversing orientations on some mated pairs of rays (with the mate determining a nonorientable end). Since the end functions are independent of i, there is a proper homotopy of $\gamma_0^\pm$ for each choice of sign, after which $\gamma_i^\pm(s, n)$ is independent of i for each $s \in S$ and $n \in \mathbb{Z}^+$. We can assume the local orientations agree at each of these points, after possibly changing the homotopy on each ray determining a nonorientable end. The proper homotopy of $\gamma_0^\pm | S \times \mathbb{Z}^+$ extends to an ambient isotopy as in the proof of Lemma 4.9, without dimensional restriction (since we only deal with the 0-skeleton Y_0). □

5. Smoothings of open 4-manifolds

Recall from Section 2 that end-summing with an exotic $\mathbb{R}^4$ can be defined as an operation on the smooth structures on a fixed topological 4-manifold, and that one can similarly change smoothings on n-manifolds by summing with an exotic $\mathbb{R} \times S^{n-1}$ along a properly embedded line. (The latter is most interesting when $n = 4$, but the comparison with higher dimensions is illuminating.) We now address uniqueness of both operations, expressing them as monoid actions on the set of isotopy classes of smoothings of a topological manifold.

We first consider end-summing with exotic $\mathbb{R}^4$'s. In [G85], it was shown that the set $\mathcal{R}$ of oriented diffeomorphism types of smooth manifolds homeomorphic to $\mathbb{R}^4$ admits the structure of a commutative monoid under end-sum, with identity $\mathbb{R}^4$, and such that countable sums are well-defined and independent of order and grouping. (The operation is defined as simultaneously end-summing onto the standard $\mathbb{R}^4$ along a multiray in the latter, so Lemma 4.9 completes the proof.) For any set S, the cartesian product $\mathcal{R}^S$ inherits a monoid structure with the same properties, as does the submonoid $\mathcal{R}^S_c$ of S-tuples that are the identity except in countably many coordinates. For any monoid $\mathcal{M}$ with such properties, we define an action on a set $\mathcal{S}$ by analogy with group actions, but allowing infinite iteration: An *action* is an assignment of a function $\mathcal{S} \to \mathcal{S}$ to each countable family $\mathcal{F}$ of elements of $\mathcal{M}$, in such a way that the same function results from any $\mathcal{F}'$ obtained by partitioning $\mathcal{F}$ into disjoint subfamilies and summing within each subfamily. (That is, the action is compatible with infinite summing in the monoid, analogously to group actions.) We also require any family consisting of just the identity to yield $\mathrm{id}_{\mathcal{S}}$. We obtain the following corollary of the lemmas of the previous section. We again split a multiray $\gamma\colon S \times [0,\infty) \to X$ into two multirays $\gamma_{\mathrm{ML}}\colon S_{\mathrm{ML}} \times [0,\infty) \to X$ and $\gamma_{\mathrm{bad}}\colon S_{\mathrm{bad}} \times [0,\infty) \to X$, according to whether each ray determines a Mittag-Leffler end.

Corollary 5.1. *Let X be a topological 4-manifold with a locally orienting multiray $\gamma\colon S \times [0,\infty) \to X$. Then γ determines an action of $\mathcal{R}^S$ on the set $\mathcal{S}(X)$ of isotopy classes of smoothings of X. The action only depends on the proper homotopy class of γ_{bad}, the function $\epsilon_{\gamma_{\mathrm{ML}}}$, and the subset of S_{ML} inducing a preassigned orientation on the orientable ends. In particular, if X is oriented (or orientations are specified on all orientable Mittag-Leffler ends) then the monoid $\mathcal{R}_c^{\mathcal{E}_{\mathrm{ML}}(X)}$ acts canonically on $\mathcal{S}(X)$.*

Note that orientation reversal induces an involution on the monoid $\mathcal{R}$, and changing the local orientations of γ changes the action by composing with this involution on the affected factors of $\mathcal{R}^S$.

Proof. To define the action, start with a smoothing on X and a family $\{R_s | s \in S\}$ of elements of $\mathcal{R}$. According to Quinn ([Q82], cf. also [FQ90]), γ can be made smooth by a topological ambient isotopy. For each $s \in S$, choose a smooth ray γ' in R_s, and use it to sum R_s with X along the corresponding ray in X. We do this by homeomorphically identifying the complement of a tubular neighborhood of γ' (with smooth $\mathbb{R}^3$ boundary) with a corresponding closed tubular neighborhood of the ray in X (preserving orientations), then transporting the smoothing of R_s to X. We assume the identification is smooth near the boundary $\mathbb{R}^3$'s, and then the smoothing fits together with the given one on the rest of X. This process can be performed simultaneously for all $s \in S$, as long as we keep the neighborhoods of the rays in X disjoint. Each ray γ' is unique up to smooth ambient isotopy (Lemma 4.9), and the required identifications of neighborhoods (homeomorphic to the half-space $[0, \infty) \times \mathbb{R}^3$) are unique up to topological ambient isotopy that is smooth on the boundary (by the Alexander trick), so the resulting isotopy class of smoothings on X is independent of choices in the R_s summands. Similarly, the resulting smoothing changes by an isotopy if the original smoothing is isotoped or γ is changed by a proper homotopy (Lemma 4.9 again). In particular, the initial choice of smoothing of γ does not matter. Since the proper homotopy class of the locally orienting multiray γ_{ML} is determined by $\epsilon_{\gamma_{\mathrm{ML}}}$ and the orientation data (Lemma 4.8(b)), we have a well-defined function $\mathcal{S}(X) \to \mathcal{S}(X)$ determined by an element of $\mathcal{R}^S$ and the data given in the corollary.

We extend to countable families from $\mathcal{R}^S$ by first summing them together in $\mathcal{R}^S$ and then using the previous procedure. That is, for each $s \in S$ we first sum the corresponding elements of $\mathcal{R}$ by end-summing them into the standard $\mathbb{R}^4$, then end-sum the result into X. Since end-summing with $\mathbb{R}^4$ is the identity operation, this procedure is equivalent to summing the elements of $\mathcal{R}$ directly into X along parallel copies of γ. We have an action as defined above since the monoid operation is independent of order and grouping (which again follows since summing with $\mathbb{R}^4$ is the identity). If we enlarge S while requiring the new manifolds R_s to all be $\mathbb{R}^4$, the induced element of $\mathcal{S}(X)$ is unchanged, so it is easy to deduce the last sentence even when $\mathcal{E}_{\mathrm{ML}}$ is uncountable. $\square$

In contrast with more general end-sums, the action of $\mathcal{R}^S$ on $\mathcal{S}(X)$ is not known to vary with the choice of proper homotopy class of γ.

Question 5.2. *Suppose that two locally orienting multirays in X have the same end function, and that for each $s \in S$, the two corresponding rays induce the same orientation on the induced end, if it admits one. Can the two actions of $\mathcal{R}^S$ on $\mathcal{S}(X)$ be different?*

We can also ask about diffeomorphism types; cf. Question 1.2. Clearly, any negative example must involve ends that fail to be Mittag-Leffler, such as those arising by ladder surgery. While such examples seem likely to exist, there are also reasons for caution, as we now discuss.

First, not all exotic $\mathbb{R}^4$'s can give such examples. Freedman and Taylor [FT86] constructed a "universal" $\mathbb{R}^4$, $R_U \in \mathcal{R}$, which is characterized as being the unique fixed point of the $\mathcal{R}$-action on $\mathbb{R}^4$. They essentially showed that for any smoothing Σ of a 4-manifold X, the result of end-summing with copies of R_U depends only on the subset of $\mathcal{E}(X)$ at which the sums are performed, regardless of whether those ends are Mittag-Leffler. Then $\mathcal{R}$ subsequently acts trivially on each of those ends. They also showed that the result of summing with R_U on a dense subset of ends creates a smoothing depending only on the stable isotopy class of Σ (classified by $H^3(X;\mathbb{Z}/2)$). For such a smoothing, $\mathcal{R}^S$ acts trivially for any choice of multiray. The main point is that the universal property is obtained through a countable collection of compact subsets of R_U that allow smooth surgery problems to be solved. If X is summed with R_U on one side of a ladder sum (for example), those compact subsets are also accessible on the other side by reaching through the rungs of the ladder.

A second issue is that negative examples would be subtle and hard to distinguish:

Proposition 5.3. *Let X be a topological 4-manifold with smoothing Σ. Let $\gamma_0, \gamma_1\colon S \times [0,\infty) \to X$ be multirays as in the above question, inducing smoothings Σ_0 and Σ_1, respectively, via a fixed element of $\mathcal{R}^S$. Then for every smooth, compact 4-manifold K, every Σ_0-smooth embedding $\iota\colon K \to X$ is topologically ambiently isotopic to a Σ_1-smooth embedding. Every neighborhood of infinity in X contains another such neighborhood U such that whenever $\iota(K) \subset U$ and K is a 2-handlebody, the resulting isotopy can be assumed to keep $\iota(K)$ inside U.*

This shows that many of the standard 4-dimensional techniques for distinguishing smooth structures will fail in the above situation. One of the oldest techniques for distinguishing two smoothings on $\mathbb{R}^4$ is to find a compact submanifold that smoothly embeds in one but not the other [G85]. A newer incarnation of this idea is the Taylor invariant [T97], distinguishing via exotic $\mathbb{R}^4$'s embedding with compact closure. Clearly, such techniques must fail in the current situation.

Most recently, the genus function has turned out to be useful [G13], distinguishing by the minimal genera of smoothly embedded surfaces representing various homology classes. However, any such surface for Σ_0 will be homologous to one of the same genus for Σ_1 and vice versa. Minimal genera at infinity [G13] will also fail: If we choose a system of neighborhoods of infinity as in the proposition, any corresponding sequence of Σ_0-smooth surfaces in these will be isotopic to a corresponding sequence for Σ_1 with the same genera. A possibility remains of distinguishing Σ_0 and Σ_1 by sequences of smoothly embedded 3-manifolds approaching infinity (such as by the engulfing index of [BG96], cf. also Remark 4.3(b) of [G13]), but there does not currently seem to be any good way to analyze such sequences. Note that the situation is not improved by passing to a cover, since the corresponding lifted smoothings will behave similarly. (The multirays γ_i will lift to multirays, cf. proof of Theorem 8.1 in [G13], and for each $s \in S$ the lifts of the corresponding rays of γ_0 and γ_1 will have end functions whose images have the same closure in $\mathcal{E}(\widetilde{X})$. The proof below still applies to this situation.)

Proof. For the first conclusion, let $\nu_i\colon S \times [0,\infty) \times D^3 \to \mathbb{R}$ be the extensions of the multirays γ_i used for the end-sums. (Recall that this variation of end-summing uses closed tubular neighborhoods.) By properness, both subsets $\nu_i^{-1}\iota(K)$ are contained in a single subset of the form $T = S_0 \times [0,n] \times D^3$ for some finite $S_0 \subset S$ and $n \in \mathbb{Z}$. For each $s \in S_0$, the corresponding rays of γ_0 and γ_1 determine the same end, and induce the same orientation on it if possible. Thus, there is a Σ-smooth ambient isotopy Φ_t with $\Phi_1 \circ \gamma_0(s,n) = \gamma_1(s,n)$ for all such s, with the local orientations agreeing at these points, and such that $\nu_1^{-1}\Phi_1\iota(K)$ still lies in T. After further isotopy, we can assume that $\Phi_1 \circ \nu_0 = \nu_1$ on T. After we perform the end-sums, the isotopy will only be topological. However, Φ_1 will be smooth on $\iota(K)$ as required, since the new smoothings correspond under Φ_1 on the images of T and the smoothing Σ is preserved elsewhere on $\iota(K)$.

Now given a neighborhood of infinity, pass to a smaller neighborhood U such that the two subsets $\nu_i^{-1}(U)$ are equal, with complement of the form $S_1 \times [0,m] \times D^3$ for some finite S_1 and $m \in \mathbb{Z}^+$. For any K and ι with $\iota(K) \subset U$, we can repeat the previous argument. There is only one difficulty: If $K = M^3 \times I$, for example, some sheets of M may be caught between ∂U and the moving image of γ_0 during the final isotopy, and be pushed out of U. However, if K is a handlebody with all indices 2 or less, we can remove the image of K from the path of γ_0 (which will be following arcs of γ_1) by transversality. The final statement now follows as before. $\square$

Next we consider sums along properly embedded lines. For a fixed $n \geq 4$, let $\mathcal{Q}$ denote the set of oriented diffeomorphism types of manifolds homeomorphic to $\mathbb{R} \times S^{n-1}$, with a given ordering of their two ends. Each such manifold admits a proper embedding of a line, preserving the order of the ends, and this is unique up to ambient isotopy by Lemma 4.9. Thus, $\mathcal{Q}$ has a well-defined commutative monoid structure induced by summing along lines, preserving orientations on the lines and n-manifolds. (This time, properness prevents infinite sums.) The identity is $\mathbb{R} \times S^{n-1}$ with its standard smoothing. For $n > 4$, $\mathcal{Q}$ is canonically isomorphic to the finite group Θ_{n-1} of homotopy $(n-1)$-spheres [KM63] (by taking their product with $\mathbb{R}$), but when $n = 4$, $\mathcal{Q}$ has much more structure. High-dimensional theory predicts that $\mathcal{Q}$ should be $\mathbb{Z}/2$, but in fact it is an uncountable monoid with an epimorphism to $\mathbb{Z}/2$ (analogous to the Rohlin invariant of homology 3-spheres). Uncountability is already suggested by Corollary 5.1, but the structure of $\mathcal{Q}$ is richer than can be obtained just by summing with exotic $\mathbb{R}^4$'s. For example, there is an uncountable subfamily in $\mathcal{Q}$ indexed by $\mathbb{R}$, with the property that for any two elements Σ and Σ', no subset of Σ that is homeomorphic to $\mathbb{R} \times S^3$ and separates the ends of Σ can smoothly embed in Σ', and vice versa. (Take the subfamily with $st = 1$ in the final remarks of [G85], Section 3.) Thus, no two elements of the family are obtained from a common third element by end-summing with exotic $\mathbb{R}^4$'s. To obtain an action on $\mathcal{S}(X)$ for $n \geq 4$, let $\gamma\colon S \times \mathbb{R} \to X$ (S discrete) be a flat, proper, locally orienting topological embedding. Then $\mathcal{Q}^S$ has a well-defined action on $\mathcal{S}(X)$ (although without infinite iteration) by the same method as before, and this only depends on the proper homotopy class of γ. (To see that a self-homeomorphism rel boundary of $\mathbb{R} \times D^{n-1}$ is isotopic to the identity, first use the topological Schoenflies Theorem to reduce to the case where $\{0\} \times D^{n-1}$ is fixed.) Note that while $\mathcal{Q}$ admits only finite sums, the set S may be countably infinite. Example 3.4 showed that the action of $\mathcal{Q}$ on $\mathcal{S}(X)$ for a two-ended 4-manifold X can depend on the choice of line spanning the ends, and in high dimensions, even the resulting diffeomorphism type can depend on the line. We next find fundamental group conditions eliminating such dependence.

To obtain such conditions, note that the fundamental progroup of X based at a ray γ has an indirect limit with well-defined image in $\pi_1(X, \gamma(0))$. (Its image equals the image of $\pi_1(U_2, \gamma(2))$ for a suitably defined neighborhood system of infinity.) If γ is instead a line, it splits as a pair $\gamma_\pm$ of rays, obtained by restricting its parameter $\pm t$ to $[0, \infty)$, determining ends $\epsilon_\pm$ and images $G_\pm \subset \pi_1(X, \gamma(0))$ of the corresponding indirect limits. We will call (ϵ_-, ϵ_+) a *Mittag-Leffler end-pair* if the double coset space $G_-\backslash\pi_1(X,\gamma(0))/G_+$ is trivial. The proof below shows that γ is then uniquely determined up to proper homotopy by the end-pair, so the condition is independent of choice of γ (as well as the direction of γ). A proper embedding $\gamma\colon\ S\times\mathbb{R}\to X$ now splits into γ_{ML} and γ_{bad} according to which lines connect Mittag-Leffler end-pairs, and the function $\epsilon_{\gamma_{\mathrm{ML}}}\colon\ S\times\{\pm1\}\to\mathcal{E}_{\mathrm{ML}}$ picks out the corresponding pairs. For simplicity, we now assume X is orientable.

Corollary 5.4. *Let X be an oriented topological n-manifold ($n\geq 4$) with a flat, proper embedding $\gamma\colon\ S\times\mathbb{R}\to X$. Then γ determines an action of $\mathcal{Q}^S$ on $\mathcal{S}(X)$. The action only depends on the proper homotopy class of γ_{bad} and the function $\epsilon_{\gamma_{\mathrm{ML}}}$.*

If X is simply connected and $\mathcal{E}_{\mathrm{ML}}$ is finite, we obtain a canonical action of $\mathcal{Q}^{\mathcal{E}_{\mathrm{ML}}\times\mathcal{E}_{\mathrm{ML}}}$ on $\mathcal{S}(X)$. In the infinite case, the situation is more complicated, since there is no proper embedding connecting all pairs of subsets (or even connecting a single end to all others).

Proof. For a proper embedding γ of $\mathbb{R}$ determining a Mittag-Leffler end-pair $\epsilon_\pm$ as above, we first show that any other embedding γ' determining the same ordered pair of ends is properly homotopic to γ. Let $\{U_i\}$ be a neighborhood system of infinity as in the proof of Lemma 4.8, and reparametrize the four rays $\gamma_\pm$ and$\gamma'_\pm$ accordingly (fixing 0). As before, we can properly homotope γ' to agree with γ on $\mathbb{Z}\subset\mathbb{R}$, so that γ and γ' are related by a doubly infinite sequence of loops. The loop captured between ±2 (starting at $\gamma(0)$, then following γ_-, γ' and (backwards) γ_+) represents a class in $\pi_1(X,\gamma(0))$ that by hypothesis can be written in the form w_-w_+ with $w_\pm\in G_\pm$. After a homotopy of γ' supported in $[-2,2]$, we can assume that $\gamma'=\gamma$ on $[-1,1]$, and the innermost loops are given by $w_\pm$ pulled back to $\pi_1(U_1,\gamma(\pm1))$. Working with each sign separately, we now complete the proof of Lemma 4.8, denoting the pullback of $w_\pm$ by x_1 as before. By the definition of $G_\pm$, x_1 can be assumed to pull back further to $\pi_1(U_2,\gamma_\pm(2))$; let y_2 be the inverse of such a pullback. Completing the construction, we see that $z_1=1$, so that γ' is then properly homotoped to γ rel $[-1,1]$.

Now that Mittag-Leffler pairs are well-defined, Corollary 5.4 follows from the discussion preceding it once we verify that γ_{ML} only affects the answer through its end function. This follows as for Lemma 4.8. □

Corollary 5.4 is most interesting when $n=4$, since classical smoothing theory reduces the higher dimensional case to discussing the Poincaré duals of the relevant lines in $H^{n-1}(X;\Theta_{n-1})$. When $n=4$, this same discussion applies to the classification of smoothings up to stable isotopy (isotopy after product with $\mathbb{R}$), by the obstruction group

$H^3(X;\mathbb{Z}/2)$, but one typically encounters uncountably many isotopy classes within each stable isotopy class. Note that the above method can be used to study sums of more general CAT manifolds along collections of lines. In dimension 4, one can also consider actions on $\mathcal{S}(X)$ of the monoid $\mathcal{Q}_k$ of oriented smooth manifolds homeomorphic to a k-punctured 4-sphere Σ_k with an order on the ends, generalizing the cases $\mathcal{Q}_1 = \mathcal{R}$ and $\mathcal{Q}_2 = \mathcal{Q}$ considered above. (The monoid operation is summing along k-fold unions of rays with a common endpoint; see the end of [G89] for a brief discussion.) However, little is known about this monoid beyond what can be deduced from Corollaries 5.1 and 5.4 and the structure of $\mathcal{R}$ and $\mathcal{Q}$. It follows formally from having infinite sums that $\mathcal{R}$ has no nontrivial invertible elements, and no nontrivial homomorphism to a group [G85]. However, the other monoids do not allow infinite sums. This leads to the following reformulation of Question 3.5:

Question 5.5. *Does $\mathcal{Q}$ (or more generally any $\mathcal{Q}_k$) have any nontrivial invertible elements? Is $H^3(\Sigma_k;\mathbb{Z}/2)$ the largest possible image of $\mathcal{Q}_k$ under a homomorphism to a group?*

6. 1-handle slides and 0/1-handle cancellation at infinity

Our uniqueness result for adding 1-handles at infinity (Theorem 4.5) readily establishes a uniqueness result for adding both 0- and 1-handles at infinity. For compact handles of index 0 and 1, one may easily construct countable sums where the results are connected and contractible, but are distinguished by their numbers of ends. In this regard, adding 0- and 1-handles at infinity turns out to be simpler. For instance, in each dimension at least four, every (at most) countable, connected, and oriented sum of 0- and 1-handles at infinity is determined by its first Betti number. As an application of this uniqueness result, we then give a very natural and partly novel proof of the hyperplane unknotting theorem. The novelty here is that 0- and 1-handles at infinity provide the basic framework in which we employ Mazur's infinite swindle.

Throughout this section, all manifolds are oriented and all handle additions respect orientations.

Let X be a not necessarily connected CAT n-manifold where $n \geq 4$. Add to X a collection of 0-handles at infinity $Z = \bigsqcup_{i\in J} z_i$ where each z_i is CAT homeomorphic to $\mathbb{R}^n$. The index set J and all others below are discrete and countable. Attach to $X \sqcup Z$ a collection of 1-handles at infinity $H = \bigsqcup_{i\in S} h_i$ where each h_i is CAT homeomorphic to $\mathbb{R}^n$ (see Figure 2). By Definition 2.1 and Theorem 4.5, H is given by multiray

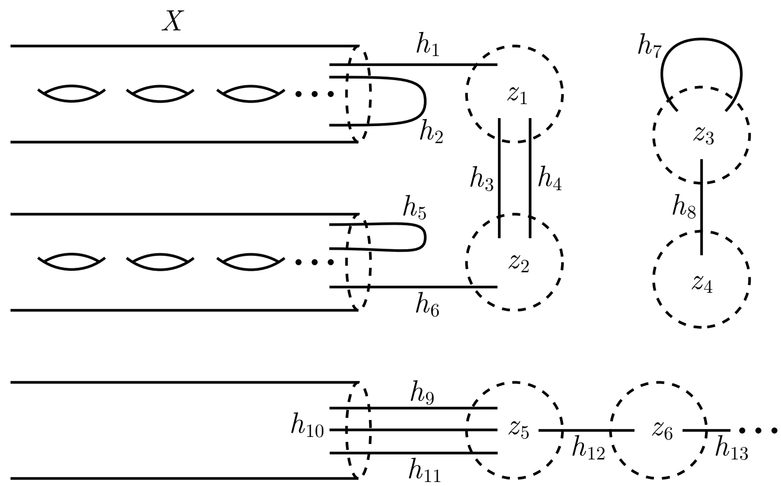

FIGURE 2. Manifold X with added 0- and 1-handles at infinity, the latter denoted by arcs.

data $\gamma^-, \gamma^+ \colon\ S \times [0,\infty) \hookrightarrow X \sqcup Z$ with disjoint images.

To this data, we associate a graph G defined as follows (see Figure 3). Let v_i, $i \in I$, be the proper homotopy classes of the rays in the multiray

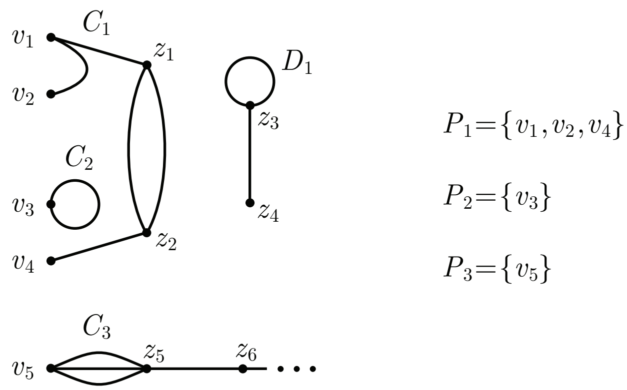

FIGURE 3. Graph G associated to the sum in Figure 2, and induced partition of the vertices v_i in X.

data for H that lie in X. Each v_i has a representative of the form $\gamma^-(j_i)$ or $\gamma^+(j_i)$ for some $j_i \in S$. The vertex set V of G is:

$$V := \{v_i \mid i \in I\} \sqcup \{z_i \mid i \in J\}\,.$$

The edges E of G are bijective with the 1-handles at infinity H and, thus, are indexed by S. The edge e_i, $i \in S$, corresponding to h_i is formally defined to be the multiset of vertices in V determined by the multiray data of h_i. In particular, E itself is a multiset, and the graph G is countable, but is not necessarily locally finite, connected, or simple. Indeed, G may have mutiple edges and loops. Let $C = \bigsqcup_{i \in I(C)} C_i$ be the connected components of G such that each component C_i contains a vertex $v_{j(i)}$ in X. Let $D = \bigsqcup_{i \in I(D)} D_i$ be the remaining components of G where each component D_i contains no vertex v_j in X. Notice that C induces a partition $\mathcal{P} = \{P_j \mid j \in I(C)\}$ of $\{v_i \mid i \in I\}$ where P_j is the subset of vertices in $\{v_i \mid i \in I\}$ that lie in C_j. Below, Betti numbers b_k are finite or countably infinite.

Theorem 6.1. *The* CAT *oriented homeomorphism type of the result Y of adding Z and H to X is determined by:*

(a) *The set of pairs $(P_j, b_1(C_j))$ where $P_j \in \mathcal{P}$.*
(b) *The multiset with elements $b_1(D_i)$ where $i \in I(D)$.*

Proof. First, consider a component D_i of G. Let M denote the component of Y corresponding to D_i. By Corollary 4.6, we can and do assume that the rays used to attach 1-handles at infinity in M are radial (while still remaining proper and disjoint). In case D_i is a tree, M is a nested union of n-disks and hence is a copy of $\mathbb{R}^n$. In general, a spanning tree T of D_i determines a copy of $\mathbb{R}^n$ in M (namely, one ignores a subset of the 1-handles at infinity). Thus, M is $\mathbb{R}^n$ with $b_1(D_i)$ 1-handles at infinity attached. By Corollary 4.6, such a manifold is determined by $b_1(D_i)$.

Second, consider a component C_j of G. Let N denote the component of Y corresponding to C_j. Let N' be the n-manifold obtained from N as follows. For each vertex v_k in C_j, introduce a 0/1-handle at infinity pair where the new 1-handle at infinity attaches to a ray in the class v_k and to a ray in the new 0-handle at infinity. Also, the 1-handles at infinity in N attached to rays in the class of v_k attach in N'_j to rays in the new 0-handle at infinity. Theorem 4.5 implies that N and N' are CAT oriented homeomorphic. The graph C'_j corresponding to N' is obtained from C_j by adding a leaf to each v_k. Let T be a spanning tree of the connected graph obtained by removing the new leaves from C'_j. Then, T determines a copy of $\mathbb{R}^n$ in N'. This exhibits N' as: the components of X containing the vertices in P_j, a single 0-handle at infinity z_0, $b_1(C_j)$ oriented 1-handles at infinity attached to z_0, and an oriented 1-handle at infinity from each $v_k \in P_j$ to z_0. □

As an application of 1-handle slides and 0/1-handle cancellation at infinity, we prove the hyperplane unknotting theorem of Cantrell [C63] and Stallings [St65].

Theorem 6.2. *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be a proper* CAT *embedding where $n \geq 4$, and let $H = f(\mathbb{R}^{n-1})$. Then, there is a* CAT *homeomorphism of $\mathbb{R}^n$ that carries H to a linear hyperplane.*

A CAT proper ray in $\mathbb{R}^k$ is *unknotted* provided there is a CAT homeomorphism of $\mathbb{R}^k$ that carries the ray to a linear ray. Recall that each CAT proper ray in $\mathbb{R}^k$, $k \geq 4$, is unknotted. For CAT=PL and CAT=DIFF, this fact follows from general position, but for CAT=TOP it is nontrivial and requires Homma's method (see [CKS12, § 7]). Thus, the following holds under the hypotheses of Theorem 6.2 by taking r to be the image under f of a linear ray in $\mathbb{R}^{n-1}$.

Ray Hypothesis. *There is a* CAT *proper ray $r \subset H$ that is unknotted in both H and $\mathbb{R}^n$ where the former means $f^{-1}(r)$ is unknotted in $\mathbb{R}^{n-1}$.*

The hyperplane H separates $\mathbb{R}^n$ into two connected components by Alexander duality. Let A' and B' denote the closures in $\mathbb{R}^n$ of these two components as in Figure 4. So, $\partial A' = H = \partial B'$, and H has a

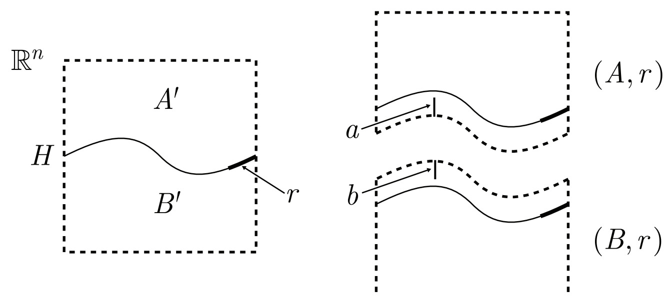

FIGURE 4. Closures A' and B' of the complement of H in $\mathbb{R}^n$ (left) and their unions A and B with open collars on H (right).

bicollar neighborhood in $\mathbb{R}^n$. Using the bicollar, define:

$$A := A' \cup (\text{open collar on } H \text{ in } B')$$
$$B := B' \cup (\text{open collar on } H \text{ in } A')$$

as in Figure 4. Figure 4 also depicts CAT proper rays $a \subset A$ and $b \subset B$ that are radial with respect to the collarings. Evidently, a and b are CAT ambient isotopic to r in A and B respectively (these simple isotopies have support in a neighborhood of the open collars).

Lemma 6.3. *It suffices to show that A' and B' are* CAT *homeomorphic to closed upper half-space $\mathbb{R}^n_+$.*

Proof. We are given CAT homeomorphisms $g : A' \to \mathbb{R}^n_+$ and $h : B' \to \mathbb{R}^n_+$. Replace h with its composition with a reflection so that $h : B' \to \mathbb{R}^n_-$. Note that g and h need not agree pointwise on H. Identify $\mathbb{R}^{n-1} \times \{0\}$ with $\mathbb{R}^{n-1}$. We have a CAT homeomorphism $j : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ given by the restriction of $g \circ h^{-1}$ to $\mathbb{R}^{n-1}$. Define the CAT homeomorphism $k : B' \to \mathbb{R}^n_-$ by $k = (j \times \mathrm{id}) \circ h$ (that is, compose h with j at each height). Now, g and k agree pointwise on H. For CAT=TOP and CAT=PL, the proof of the lemma is complete. For CAT=DIFF, one smooths along collars as in Hirsch [H94, Thm. 1.9, p. 182]. □

We will use the symbols in Figure 5 to denote the indicated manifold/ray pairs. Here, c is a radial ray in $\mathbb{R}^n$. All rays in this proof will

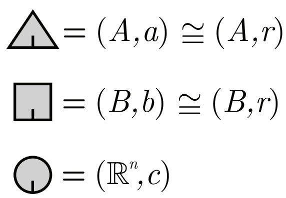

FIGURE 5. Notation for relevant manifold/ray pairs.

be parallel (CAT ambient isotopic) to r or c (these include a and b). An added 1-handle at infinity will be denoted by an arc connecting such symbols as in Figure 6.

Lemma 6.4. *All three of the manifold/ray pairs in Figure 6 are* CAT *homeomorphic to one another.*

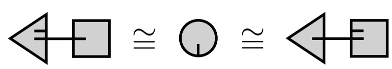

FIGURE 6. Isomorphic manifold/ray pairs.

Proof. First, adding a 1-handle at infinity to $(A, a) \sqcup (B, b)$ yields $\mathbb{R}^n$. To see this, recall Figure 4 and choose the tubular neighborhood maps for the 1-handle at infinity to be the full collars in the $\mathbb{R}^{n-1}$ directions. Next, let a' and b' be the indicated rays in Figure 6 parallel to a and b respectively. The lemma follows by shrinking the above tubular neighborhood maps in the $\mathbb{R}^{n-1}$ directions to be disjoint from a' and b' respectively. □

Lemma 6.5. *It suffices to prove that* (A, a) *and* (B, b) *are* CAT *homeomorphic as pairs to* $(\mathbb{R}^n, c)$.

Proof. First, consider the cases CAT=DIFF and CAT=PL. The collar on H in A is a CAT *closed* regular neighborhood of a in A with boundary H. Using the hypothesis $(A, a) \cong (\mathbb{R}^n, c)$, apply uniqueness of such neighborhoods in $(\mathbb{R}^n, c)$ to see that A' is CAT homeomorphic to $\mathbb{R}^n_+$. Similarly, we get B' is CAT homeomorphic to $\mathbb{R}^n_+$. Now, apply Lemma 6.3.

For CAT=TOP, we are given a homeomorphism $g : (A, a) \to (\mathbb{R}^n, c)$. Let $V \cong \mathbb{R}^n_+$ be the collar added to A' along H to obtain A as in Figure 4. Let $U \cong \mathbb{R}^n_+$ be a collar on H in A on the opposite side of H as in Figure 7. Recall that $\mathbb{R}^n$ itself is an open mapping cylinder

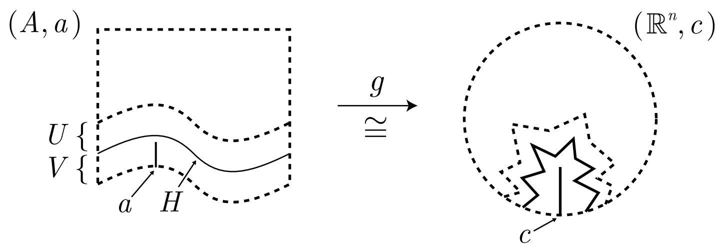

FIGURE 7. Homeomorphic manifold/ray pairs (A, a) and $(\mathbb{R}^n, c)$. Also depicted are the hyperplane H, the collar V added to A' to obtain A, a collar U on the other side of H, and their images in $\mathbb{R}^n$.

neighborhood of c in $\mathbb{R}^n$ (see [KR63] and [CKS12, pp. 1816,1831]). Similarly, $U \cup V$ is an open mapping cylinder neighborhood of a in $U \cup V$. So, $g(U \cup V)$ is another open mapping cylinder neighborhood of c in $\mathbb{R}^n$. Uniqueness of such neighborhoods (see [KR63] and [CKS12]) implies there exists a homeomorphism $h : g(U \cup V) \to \mathbb{R}^n$ that fixes $g(V)$ pointwise. Therefore:

$$g(U) \cong \mathbb{R}^n - \mathrm{Int}g(V) = g(A').$$

Hence, $A' \cong U \cong \mathbb{R}^n_+$. Similarly, B' is homeomorphic to $\mathbb{R}^n_+$. Again, Lemma 6.3 completes the proof. □

Finally, we come to the heart of the proof of the hyperplane unknotting theorem. Mazur's infinite swindle [M59] is realized as 1-handle slides and 0/1-handle cancellations at infinity. Figure 8 proves that (A, a) is CAT homeomorphic to $(\mathbb{R}^n, c)$. In Figure 8, the horizontal region is a copy of $\mathbb{R}^n$. The first, third, and fifth isomorphisms in Figure 8 hold by Theorem 6.1. The second and fourth isomorphisms hold by Lemma 6.4. With $(A, a) \cong (\mathbb{R}^n, c)$, Figure 6 implies that

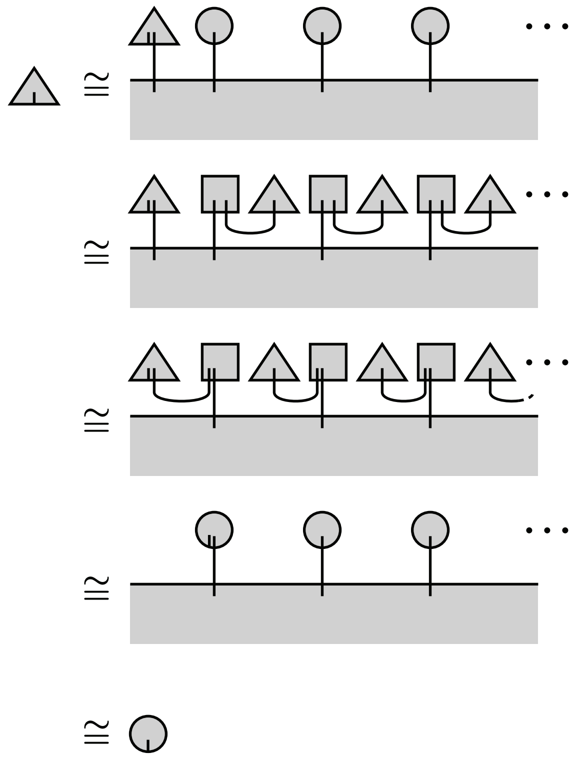

FIGURE 8. Mazur's infinite swindle as 1-handle slides and 0/1-handle cancellations at infinity.

$(B, b) \cong (\mathbb{R}^n, c)$. By Lemma 6.5, our proof of the hyperplane unknotting theorem is complete.

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