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VIETORIS-BEGLE THEOREM AND SPECTRA

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ABSTRACT. The following generalization of the Vietoris-Begle Theorem is proved: Suppose $\{E_k\}_{k\geq 1}$ is a CW spectrum and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that $\operatorname{Ind} X = m < \infty$. If $f^*: E^k(x) \to E^k(f^{-1}(x))$ is an isomorphism for all $x \in X$ and $k = m_0$, $m_0 + 1, \ldots, m_0 + m$, then $f^*: E^n(X) \to E^n(X')$ is an isomorphism and $f^*: E^{n+1}(X) \to E^{n+1}(X')$ is a monomorphism for $n = m_0 + m$.

Given a CW spectrum $E = \{E_k\}_{k\geq 1}$ and a pointed CW complex K, one has cohomology groups $E^n(K)$ for each integer n (see [Sw, Chapter 8]). They are defined as homotopy classes from the suspension spectrum of K to $\Sigma^n E$, where $\Sigma^n E$ is defined by $\Sigma^n E_k = E_{k+n}$. In the case of an Ω-spectrum (i.e., where the natural map $E_k \to \Omega E_{k+1}$ is a homotopy equivalence for all k), $E^n(K)$ is isomorphic to $[K, E_n]$ (see [Sw, Theorem 8.42]). The groups $E^n(X)$, X being any pointed topological space, are defined as $\operatorname{dir}\lim\{E^n(X_\alpha), p^*_{\alpha\beta}, \Lambda\}$, where $\{X_\alpha, p_{\alpha\beta}, \Lambda\}$ is the Čech system of X (see [D-S, p. 21] for the definition of the Čech system of X). In this way one gets the Čech extension of the functor E^n from pointed CW complexes to pointed spaces (see [D] for a general discussion of Čech extensions of functors). Again, if $\{E_k\}_{k\geq 1}$ is an Ω-spectrum, then $E^n(X)$ is isomorphic to $[X, E_n]$. A basic result is that every spectrum $\{E_k\}_{k\geq 1}$ is isomorphic to an Ω-spectrum (see [B, part 10 of Chapter II]). Essentially, the nth term of that spectrum is the telescope of $E_n \to \Omega E_{n+1} \to \Omega^n E_{n+2} \to \cdots$.

In the case of an unpointed topological space X, we define the unreduced cohomology $E^n(X)$ as $E^n(X^+)$, where X^+ is X with a discrete base point added.

Recall the classical Vietoris-Begle Theorem (see [S, p. 344]):

Vietoris-Begle Theorem. *Let $f: X' \to X$ be a closed surjective map of paracompact Hausdorff spaces. Assume that there is an $n \geq 0$ such that $\tilde{H}^k(f^{-1}(x)) = 0$ (reduced Čech cohomology) for all $x \in X$ and for $k < n$. Then $f^*: H^k(X) \to H^k(X')$ is an isomorphism for $k < n$ and a monomorphism for $k = n$.*

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A natural question arises: Since Čech cohomology corresponds to the Eilenberg-MacLane spectrum $\{K(G, k)\}_{k\geq 1}$, is there a generalization of the above theorem to arbitrary spectra?

The naive approach of replacing $\widetilde{H}^k(f^{-1}(x)) = 0$ by $E^k(f^{-1}(x)) \approx E^k(\{x\})$ does not work. An example of that is Taylor's cell-like map (see [T]) $f: X \to Q$ onto the Hilbert cube Q such that $\widetilde{K}(X)$ is nonzero ($\widetilde{K}$ is the reduced complex K theory which is the cohomology theory of the spectrum **BU**, the periodic spectrum $U, BU, \ldots$). This example was later modified by J. Keesling [K] who produced a cell-like map $g: Q \to Y$ with $\widetilde{K}(Y) \neq 0$.

The aim of this paper is to prove suitable generalizations of the Vietoris-Begle Theorem involving arbitrary unreduced spectral cohomology. In this task we were guided by Kozlowski's result (see [Ko] or [D-S]) proved in 1975.

Theorem (G. Kozlowski). *For closed maps $f: X \to Y$ of metrizable spaces such that $[f^{-1}(A), K] = [A, K]$ for any* CW *complex K and any closed subset A of Y, the image Y is an* ANR *provided X as an* ANR.

Kozlowski's proof involved a trick: he showed that certain maps from X to K can be extended over the mapping cylinder $M(f)$ of f. Then he replaced X by the double mapping cylinder $\mathrm{DM}(f)$ of f (the union of two copies of $M(f)$ sewn along X) and used the previous construction to relate any two different extensions. The meaning of this trick is that it echoes the Mayer-Vietoris exact sequence (once you prove that a certain homomorphism is onto you get that the next one is trivial, for free). In this paper we formalize this observation: the trick becomes Lemma 3 (the mapping cone C_p of $p: \mathrm{DM}(f) \to Y$ is homeomorphic to the reduced suspension $S(C_f)$ of the mapping cone of f) and the whole approach resembles a Puppe exact sequence. Notice that Theorem B implies the results of [D-K] (the goal of that paper was to unify Vietoris-Begle Theorem and cell-like maps on spaces of finite deformation dimension).

Theorem A. *Suppose $\{E_k\}_{k\geq 1}$ is a* CW *spectrum and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that* $\operatorname{Ind} X = m < \infty$. *If $f^*: E^n(x) \to E^n(f^{-1}(x))$ is an isomorphism for all $x \in X$ and $n = m_0, m_0+1, \ldots, m_0+m$, then $f^*: E^n(X) \to E^n(X')$ is an isomorphism and $f^*: E^{n+1}(X) \to E^{n+1}(X')$ is a monomorphism for $n = m_0 + m$.*

Remark. $\operatorname{Ind} X$ is the *large inductive dimension* of X: $\operatorname{Ind}\varnothing = -1$ and $\operatorname{Ind} X \leq m$ means that for any neighborhood U of a closed subset A of X, there is a neighborhood V of A in U with $\operatorname{Ind}(\mathrm{cl}(V) - V) \leq m - 1$.

Theorem B. *Suppose $\{E_k\}_{k\geq 1}$ is a* CW *spectrum and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that the following conditions are satisfied*:

(a) $f^*: E^{n-1}(A) \to E^{n-1}(f^{-1}(A))$ *is an epimorphism for all closed subsets A of X, and*

(b) $f^*: E^n(A) \to E^n(f^{-1}(A))$ *is a monomorphism for all closed subsets* A *of* X.

If $f^*: E^n(x) \to E^n(f^{-1}(x))$ *is an isomorphism for all* $x \in X$, *then* $f^*: E^n(X) \to E^n(X')$ *is an isomorphism and* $f^*: E^{n+1}(X) \to E^{n+1}(X')$ *is a monomorphism.*

Remark. Obviously, conditions (a) and (b) are derived from Kozlowski's Theorem.

The proofs of Theorems A and B will depend on Lemmas 1–4 below.

Given a map $f: X' \to X$ and a subset A of X, $f^{-1}(A)$ is denoted by A' and the map $A' \to A$ defined by f is denoted by f_A.

Let $C_f = M(f)/X'$ be the mapping cone of a map $f: X' \to X$. $q_f: M(f) \to C_f$ denotes the quotient map from the mapping cylinder $M(f)$ of f to C_f. Given a map $f: X' \to X$ and a space E, $f^*: [X, E] \to [X', E]$ is called *monic* provided that for any map $g: X \to E$, $gf \approx$ const implies $g \approx$ const.

Lemma 1. *Suppose* $f: X' \to X$ *is a map such that* $f^*: [X, \Omega E] \to [X', \Omega E]$ *is onto. Then* $q_f^*: [C_f, E] \to [M(f), E]$ *is monic.*

Proof. Suppose $g: C_f \to E$ is a map such that $g|X \approx$ const. We may assume $g|X =$ const (by homotoping g). Then g factors as $C_f \to \Sigma X' \to E$, which in turn factors (up to homotopy) as $C_f \to \Sigma X' \to \Sigma X \to E$. Notice that $C_f \to \Sigma X$ is null-homotopic, as it factors as $C_f \to C(X) \to \Sigma X$, $C(X)$ being the cone over X. Thus $g \approx$ const.

Lemma 2. *Suppose* $f: X' \to X$ *is a map such that* $q_f^*: [C_f, E] \to [M(f), E]$ *is monic. If* $g, h: M(f) \to E$ *are two null-homotopic maps such that* $g|X' = h|X'$, *then* $g \approx h$ rel. X'.

Proof. We need to extend the map $G: M(f) \times \{0, 1\} \cup X' \times I \to E$, where $G|M(f)\times\{0\} = g$, $G|M(f)\times\{1\} = h$, and $G(x, t) = g(x)$ for $(x, t) \in X'\times I$, over $M(f)\times I$. Since $G|X \times \{0\} \approx$ const, G extends to $G': (M(f) \times \{0, 1\} \cup X' \times I) \cup C(X \times \{0\}) \to E$, where $C(X \times \{0\})$ is the cone over $X \times \{0\}$. Notice that $M(f) \times \{0, 1\} \cup X' \times I) \cup C(X \times \{0\}$ is homotopy equivalent to C_f and $G'|X \times \{1\}$ is null-homotopic. By the hypotheses, $G' \approx$ const, which implies $G \approx$ const. Since the pair $(M(f)\times I, M(f)\times\{0, 1\}\cup X'\times I)$ has the homotopy extension property with respect to any space, we obtain an extension of G over $M(f) \times I$.

Recall that the double mapping cylinder $\mathrm{DM}(f)$ of a map $f: X' \to X$ is the union of two copies of $M(f)$ with two copies of X' identified. The natural projection $\mathrm{DM}(f) \to X$ is denoted by p.

Lemma 3. *For any map* $f: X' \to X$, *the mapping cone* C_p *of the natural projection* $p: \mathrm{DM}(f) \to X$ *is homeomorphic to the reduced suspension* $S(C_f)$ *of the mapping cone of* f.

Proof. Notice that $\mathrm{DM}(f)$ is homeomorphic to $X' \times I \cup M(f) \times \{0, 1\} \subset M(f) \times I$, and $M(p)$ is homeomorphic to $M(f) \times I$. Also C_p is homeomorphic to $M(f) \times I/(X' \times I \cup M(f) \times \{0, 1\})$. Since $\Sigma(C_f) = (M(f)/X') \times I/(M(f)/X') \times \{0, 1\}$, Lemma 3 follows.

Lemma 4. *Suppose E is a* CW *complex and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces. Denote by $\mathcal{S}$ the family of all closed subsets B of X such that $q_{f_A}^*: [C_{f_A}, E] \to [M(f_A), E]$ and $f_A^*: [A, E] \to [f^{-1}(A), E]$ are monic for any closed subset A of B. If for any closed subset B of X and for any neighborhood U of B there is an open neighborhood V of B in U such that* $\mathrm{cl}(V) - V \in \mathcal{S}$, *then the image of $f^*: [X, E] \to [X', E]$ is precisely the set of all homotopy classes $[g]$ such that $g|f^{-1}(x) \approx$* const *for all $x \in X$.*

Proof. It suffices to show that any map $g: X' \to E$ such that $g|f^{-1}(x) \approx \mathrm{const}$ for all $x \in X$ extends over $M(f)$. Without loss of generality we may assume that E is an ANE for paracompact spaces (see [D-K]). Let $\pi: M(f) \to X$ be the projection. Fix $x \in X$. Since $g|f^{-1}(x) \approx \mathrm{const}$, there exists an extension $g': X' \cup \pi^{-1}(x) \to E$ of g. Define $g'': X' \cup \pi^{-1}(x) \cup X \to E$ by $g''|X' \cup \pi^{-1}(x) = g'$ and $g''(X) = g'(x)$; g'' extends over a neighborhood U of $X' \cup \pi^{-1}(x) \cup X$ in $M(f)$. Choose a neighborhood V_x of x in X such that $\pi^{-1}(V_x) \subset U$. Having done that for all x in X, we choose a locally finite cover $\{A_s: s \in S\}$ of X consisting of closed sets, which is a refinement of $\{V_x: x \in X\}$. Then, for each $s \in S$, we choose a map $g_s: X' \cup \pi^{-1}(A_s) \to E$ such that $g_s|X' = g$ and $g_s(A_s)$ is a one-point set.

If $g': X' \cup \pi^{-1}(A) \to E$ is an extension of g (A closed in X) and $s \in S$, then there is an extension g'' of g' over $X' \cup \pi^{-1}(U)$ for some closed neighborhood U of A. Choose an open neighborhood V of A in U such that $\mathrm{cl}(V) - V \in \mathcal{S}$. By Lemma 2,

$$g''|\pi^{-1}((\mathrm{cl}(V) - V) \cap A_s) \approx g_s|\pi^{-1}((\mathrm{cl}(V) - V) \cap A_s)$$

rel. $f^{-1}((\mathrm{cl}(V)-V) \cap A_s)$. Since $g_s|\pi^{-1}((\mathrm{cl}(V)-V) \cap A_s)$ extends over $\pi^{-1}(A_s)$, g'' extends over $\pi^{-1}(A_s)$. Thus we have an extension $g''': X' \cup \pi^{-1}(A \cup A_s) \to E$ of g. By well-ordering S and transfinite induction, we can construct an extension $G: M(f) \to E$ of g.

Proof of Theorems A *and* B. We are going to prove the following statement which implies both Theorems A and B:

($*$) Suppose $\{E_k\}_{k \geq 1}$ is a CW spectrum and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that, for some integer n, $f^*: E^n(x) \to E^n(f^{-1}(x))$ is an isomorphism for all $x \in X$. Denote by $\mathcal{S}$ the family of all closed subsets B of X such that $f^*: E^{n-1}(A) \to E^{n-1}(f^{-1}(A))$ is an epimorphism, and $f^*: E^n(A) \to E^n(f^{-1}(A))$ is a monomorphism for all closed subsets A of B. If for any closed subset B of X and for any neighborhood U of B there is an open neighborhood V of B in U such that $\mathrm{cl}(V) - V \in \mathcal{S}$, then $f^*: E^n(X) \to E^n(X')$ is an isomorphism and $f^*: E^{n+1}(X) \to E^{n+1}(X')$ is a monomorphism.

Without loss of generality, assume $\{E_k\}_{k\geq 1}$ is an Ω-spectrum. Now, $E^n(Z) = [Z, E_n]$ for any space Z.

Notice that $f^*: E^n(X) \to E^n(X')$ is an isomorphism by Lemma 4. Indeed, $f_A^*: [A, E_n] \to [f^{-1}(A), E_n]$ is monic for each $A \in \mathcal{S}$, and Lemma 1 implies that $q_{f_A}^*: [C_{f_A}, E_n] \to [M(f_A), E_n]$ is monic for each $A \in \mathcal{S}$.

So it remains to show that $f^*: E^{n+1}(X) \to E^{n+1}(X')$ is a monomorphism. Suppose $g, h: X \to E_{n+1}$ are two maps such that $gf \approx hf$. Then there is a map $H: \mathrm{DM}(f) \to E_{n+1}$ such that H restricted to one copy of X equals g and H restricted to the other copy of X equals h. It suffices to show that H extends over $M(p)$, where $p: \mathrm{DM}(f) \to X$ is the natural projection. This is easily seen if one notices that $(M(p), \mathrm{DM}(f))$ is homeomorphic to $(M(f) \times I, X' \times I \cup M(f) \times \{0, 1\})$. Then $H|X \times \{0\} = g$, $H|X \times \{1\} = h$, and any extension of H over $M(f) \times I$ would produce a homotopy from g to h when restricted to $X \times I$.

Notice that $H|p^{-1}(x)$ is null-homotopic for all $x \in X$. Indeed, $p^{-1}(x)$ is the suspension $\Sigma f^{-1}(x)$ of $f^{-1}(x)$ and $[\Sigma f^{-1}(x), E_{n+1}] = [f^{-1}(x), \Omega E_{n+1}] = [f^{-1}(x), E_n] = [\{x\}, E_n]$. To be able to apply Lemma 4, we need to check that for each $A \in \mathcal{S}$, both $q_{p_A}^*: [C_{p_A}, E_{n+1}] \to [M(p_A), E_{n+1}]$ and $p_A^*: [A, E_{n+1}] \to [p^{-1}(A), E_{n+1}]$ are monic. The latter is clear, since p is a retraction. By Lemma 3, $[C_{p_A}, E_{n+1}] = [S(C_{f_A}), E_{n+1}] = [C_{f_A}, \Omega E_{n+1}] = [C_{f_A}, E_n]$ (here C_{p_A} and C_{f_A} are considered as pointed spaces with obvious base points). The proof can be completed by showing that any map u from C_{f_A} to E_n is null-homotopic (this implies that all pointed maps from C_{p_A} to E_{n+1} are null-homotopic, and since all components of $E_{n+1} \cong \Omega E_{n+2}$ are of the same homotopy type, all unpointed maps from C_{p_A} to E_{n+1} are null-homotopic). Since $(u|A)f \approx \mathrm{const}$, $u|A \approx \mathrm{const}$ by the first part of the Theorem. Thus u factors (up to homotopy) as $C_{f_A} \to C_{f_A}/A = \Sigma f^{-1}(A) \to E_n$. Since $f^*: E^{n-1}(A) \to E^{n-1}(f^{-1}(A))$ is an epimorphism, the map $\Sigma f^{-1}(A) \to E_n$ factors (up to homotopy) as $\Sigma f^{-1}(A) \to \Sigma(A) \to E_n$. Since $C_{f_A} \to \Sigma(A)$ factors through the cone over A, $u \approx \mathrm{const}$.

Statement $(*)$ obviously implies Theorem B. Theorem A can be proved by induction on $m = \mathrm{Ind}\, X$. If $m = 0$, we use Statement $(*)$ with $\mathcal{S}$ being empty and $n = m_0$. If Theorem A holds for $m \leq k$ and $\mathrm{Ind}\, X = k + 1$, we use Statement $(*)$ with $\mathcal{S} = \{B \in \mathrm{cl}(B) \subset X | \mathrm{Ind}\, B \leq k\}$ and $n = k + 1 + m_0$.

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