Topology and its Applications 23 (1986) 75-86
North-Holland

AN ADDENDUM TO THE VIETORIS–BEGLE THEOREM

Jerzy DYDAK
Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, U.S.A.

Received 11 March 1985
Revised 21 January 1986

The classical Vietoris–Begle Theorem is improved by observing that the image of $\bar{H}^n(f)$: $\bar{H}(Y; G) \to \bar{H}^n(X: G)$ is $\bigcap_{y\in Y} \ker(\bar{H}^n(X; G) \to \bar{H}^n(f^{-1}(y); G))$. This implies the Dual Vietoris–Begle Theorem for Steenrod homology and pro-homology, in which case one can characterize the kernel of the induced homomorphism.

AMS (MOS) Subj. Class.: Primary 55N07; Secondary 55N05

Vietoris–Begle Theorem — Steenrod homology
Alexander–Spanier Cohomology — pro-homology

Introduction

First recall the famous

Vietoris–Begle theorem. *Let f: $X \to Y$ be a closed continuous surjective map between paracompact Hausdorff spaces. Assume that there is $n \geq 0$ such that $\bar{H}^k(f^{-1}(y); G) = 0$ for all $y \in Y$ and for $k < n$. Then*

$$\bar{H}^k(f)\colon\ \bar{H}^k(Y; G) \to \bar{H}^k(X; G)$$

is an isomorphism for $k < n$ and a monomorphism for $k = n$.

Here $\bar{H}^k(A; G)$ denotes the Alexander–Spanier cohomology of A with coefficients in a group G. For a proof see [15, p. 344] or [10]. Recently the following dual to the above theorem was proved by Volovikov and Nguen [16].

Dual to Vietoris–Begle theorem. *Let f: $X \to Y$ be a surjective map between compact metrizable spaces. Assume that there is $n \geq 0$ such that $\bar{H}_k(f^{-1}(y); \mathbb{Z}) = 0$ for all $y \in Y$ and for $k < n$. Then*

$$\bar{H}_k(f)\colon\ \bar{H}_k(X; \mathbb{Z}) \to \bar{H}_k(Y: \mathbb{Z})$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$.

0166-8641/86/$3.50

Their proof relies on the Leray spectral sequence (see [1])

The aim of this paper is to improve the Vietoris-Begle Theorem by stating that the image of $\tilde{\bar{H}}^n(f)$: $\tilde{\bar{H}}^n(Y: G) \to \tilde{\bar{H}}^n(X: G)$ is $\bigcap_{y \in Y} \ker(\tilde{\bar{H}}^n(X; G) \to \tilde{\bar{H}}^n(f^{-1}(y))$. Using this characterization of im $\tilde{\bar{H}}^n(f)$ we show that the Dual Vietoris-Begle Theorem is a consequence of the Vietoris-Begle Theorem. Then, as in the case of cohomology, we are able to shed light on the kernel of $\tilde{\bar{H}}_n(f)$: $\tilde{\bar{H}}_n(X; \mathbb{Z}) \to \tilde{\bar{H}}_n(Y: \mathbb{Z})$.

In the last part of the paper analogous results are proved for pro-homology and Čech homology.

Here $\bar{H}_n(A: \mathbb{Z})$ denotes the Alexander-Spanier (or Steenrod) homology of A with coefficients in the group of integers $\mathbb{Z}$ (see [11]).

If $A \subset B$ and $\alpha \in \bar{H}^k(B; G)$, then α/A denotes the image of α by the homomorphism $\bar{H}^k(B; G) \to \bar{H}^k(A; G)$ induced by the inclusion.

By the reduced homology $\tilde{\bar{H}}_k(A; G)$ (reduced cohomology $\tilde{\bar{H}}^k(A; G)$) we mean the kernel of $\bar{H}_k(A; G) \to \bar{H}_k(pt; G)$ (the cokernel of $\bar{H}^k(pt; G) \to \bar{H}^k(A; G)$). In both cases the homomorphism are induced by the projection $A \to pt$.

When using integral coefficients we will employ the notation $\bar{H}_n(A)$ for $\bar{H}_n(A; \mathbb{Z})$.

1. Improved Vietoris–Begle theorem

Theorem 1. *Let $f: X \to Y$ be a closed surjective map between paracompact Hausdorff spaces. Assume that there is $n \geq 0$ such that $\tilde{\bar{H}}^k(f^{-1}(y); G) = 0$ for all $y \in Y$ and for $k < n$. Then the sequence*

$$0 \to \tilde{\bar{H}}^k(Y; G) \xrightarrow{\tilde{\bar{H}}^k(f)} \tilde{\bar{H}}^k(X; G) \xrightarrow{\gamma} \prod_{y \in Y} \tilde{\bar{H}}^k(f^{-1}(y); G)$$

is exact for $k \leq n$ where γ is induced by inclusion induced homomorphisms $\tilde{\bar{H}}^k(X; G) \to \tilde{\bar{H}}^k(f^{-1}(y); G)$.

Proof. It suffices to show that for each $\alpha \in \bar{H}^n(X; G)$ such that $\alpha/f^{-1}(y) = 0$ for all $y \in Y$ there is $\beta \in \bar{H}^n(Y; G)$ with $\alpha = \bar{H}^n(f)(\beta)$. This follows from a technique used by Lawson [10] in the following way:

Let U be the family of all closed sets V of Y such that $\alpha/f^{-1}(V) = \bar{H}^n(f/f^{-1}(V))(\beta_V)$ for some $\beta_V \in \bar{H}^n(V; G)$. In order to show that $Y \in U$ it suffices to check the following properties of U (see [12]):

(1) For each point $y \in Y$ there is $V \in U$ containing y in its interior.

(2) V, $W \in U$ implies $V \cup W \in U$.

(3) $V_S \in U$ for $s \in S$ and $\{V_S\}_{s \in S}$ is discrete implies $\bigcup_{s \in S} V_S \in U$.

(4) If $V \in U$ then each closed subset of V belongs to U.

If $B \subset Y$, then B' denotes $f^{-1}(B)$.

Property 1 follows from the fact that

$$\bar{H}^n(f^{-1}(y); G) = \varinjlim \{\bar{H}^n(V'; G);\ y \in \operatorname{int} V\}$$

(see [15, p. 316]). Therefore $\alpha/V'=0$ for some closed neighborhood V of y and we can choose $\beta_V=0$.

Property 2 follows from the naturality of the Mayer–Vietoris sequence (see [11, p. 246]):

$$\begin{array}{ccccccc}
\bar{H}^{n-1}(V\cap W) & \xrightarrow{h} & \bar{H}^n(V\cup W) & \longrightarrow & \bar{H}^n(V)\oplus\bar{H}^n(W) & \xrightarrow{b} & \bar{H}^n(V\cap W)\\
\downarrow g & & \downarrow & & \downarrow d & & \downarrow a\\
\bar{H}^{n-1}(V'\cap W') & \xrightarrow{e} & \bar{H}^n(V'\cup W') & \xrightarrow{c} & \bar{H}^n(V')\oplus\bar{H}^n(W') & \longrightarrow & \bar{H}^n(V'\cap W')
\end{array}$$

Since a is a monomorphism we have $b(\beta_V,\beta_W)=0$ which implies the existence of $\omega\in\bar{H}^n(V\cup W)$ with $\omega/V=\beta_V$, $\omega/W=\beta_W$. Then $c(\alpha/V'\cup W'-f^*(\omega))=0$ because d is a monomorphism. So $\alpha/V'\cup W'-f^*(\omega)=e(\eta)$ for some $\eta\in\bar{H}^{n-1}(V'\cap W')$ and $\eta=f^*(\mu)$ because g is an isomorphism. Then put $\beta_{V\cup W}=\omega+h(\mu)$.

Property 3 follows from the fact that $\bar{H}^n(\bigcup_{s\in S}V_S;G)=\prod_{s\in S}\bar{H}^n(V_S;G)$ and Property 4 is a consequence of the fact that we can put $\beta_W=\beta_V/W$ for $W\subset V$. □

As an immediate consequence of Theorem 1 we have

Corollary 1. *Let $f:X\to Y$ be a closed surjective map between paracompact Hausdorff spaces. Assume that there is $n\geq 0$ such that $\tilde{\bar{H}}^k(f^{-1}(y);G)=0$ for all $y\in Y$ and for $k<n$. If the inclusion induced homomorphism*

$$\tilde{\bar{H}}^n(X;G)\to\tilde{\bar{H}}^n(f^{-1}(y);G)$$

is trivial for all $y\in Y$, then

$$\bar{H}^n(f):\ \bar{H}^n(Y;G)\to\bar{H}^n(X;G)$$

is an isomorphism.

Corollary 2. *Let R be a PID (principle ideal domain) and let $f:X\to Y$ be a surjective map between compact metrizable spaces such that for all $y\in Y$, $\tilde{\bar{H}}^k(f^{-1}(y);R)$ is trivial for $k<n$ $(n\geq 0)$ and is torsion free for $k=n$. Then $\bar{H}^n(M(f),X;R)$ and the cokernel of $\bar{H}^n(f)$: $\bar{H}^n(Y;R)\to H^n(X;R)$ are torsion-free. If $R=\mathbb{Z}$ and $\bar{H}^n(f^{-1}(y);\mathbb{Z})$ is free Abelian for all $y\in Y$, then $\bar{H}^{n+1}(M(f),X;\mathbb{Z})$ and the cokernel of $\bar{H}^n(f)$ are free Abelian.*

Proof. Since $\prod_{y\in Y}\tilde{\bar{H}}^n(f^{-1}(y);R)$ is torsion free, the cokernel of $\tilde{\bar{H}}^n(f)$ is torsion-free. If $R=\mathbb{Z}$ and $\bar{H}^n(f^{-1}(y);\mathbb{Z})$ are free, $\prod_{y\in Y}\tilde{\bar{H}}^n(f^{-1}(y);\mathbb{Z})$ can be embedded as a subgroup of $\prod_{s\in S}\mathbb{Z}_s$, where each $\mathbb{Z}_s$ is a copy of $\mathbb{Z}$. Therefore, $\mathrm{im}(\gamma)$ can be embedded in $\prod_{s\in S}\mathbb{Z}_s$ and since $\bar{H}^n(X)$ is countable, $\mathrm{im}(\gamma)$ is countable, too. By

Theorem 19.2 in [6] each countable subgroup of $\prod_{s\in S}\mathbb{Z}_s$ is free Abelian and consequently the cokernel of $\bar{H}^n(f)$ is free Abelian.

To prove the remaining parts of Corollary 2 consider the double mapping cylinder DM of f, that is $\mathrm{DM}=X\times[-1,1]\bigcup_\alpha Y\times\{-1,1\}$, where $\alpha: X\times\{-1,1\}\to Y\times\{-1,1\}$ is defined by $\alpha(x,t)=(f(x),t)$. The natural projection from DM onto Y is denoted by p, $i: Y\to \mathrm{DM}$ is defined by $i(y)=(y,1)$. Then $p\cdot i=\mathrm{id}_Y$ and $p^{-1}(y)$ is the suspension of $f^{-1}(y)$ for each $y\in Y$. By what we have already proved the cokernel G of $\bar{H}^{n+1}(p)$ is torsionfree (free Abelian in the other case). Since $p\cdot i=\mathrm{id}_Y$, $\bar{H}^{n+1}(\mathrm{DM})=G\oplus\bar{H}^{n+1}(Y)$ and from the exact sequence

$$0\to\bar{H}^{n+1}(\mathrm{DM},Y)\to\bar{H}^{n+1}(\mathrm{DM})\to\bar{H}^{n+1}(Y)$$

(here Y is identified with $Y\times\{1\}$ via i) we infer that $\bar{H}^{n+1}(\mathrm{DM},Y)$ is a subgroup of G. Thus $\bar{H}^{n+1}(\mathrm{DM},Y)$ is torsionfree (free Abelian).

Since DM/Y is homeomorphic to $M(f)/X$, where $M(f)$ is the mapping cylinder of f, $\bar{H}^{n+1}(M(f),X)$ is isomorphic to $\bar{H}^{n+1}(\mathrm{DM},Y)$ (see [11, p. 17] or [15, Lemma 11]). Thus $\bar{H}^{n+1}(M(f),X)$ is torsionfree (free Abelian). □

2. Dual results

Proof of the Dual Vietoris–Begle theorem. Using the short exact sequence (see [11, p. 109])

$$0\to\mathrm{Ext}(\bar{H}^k(B,A),\mathbb{Z})\to\bar{H}_{k-1}(B,A)\to\mathrm{Hom}(\bar{H}^{k-1}(B,A),Z)\to 0$$

we get that $\bar{H}^k(f^{-1}(y);\mathbb{Z})$ is trivial for $k<n$ and free Abelian for $k=n$ (see [8, p. 107]). By Corollary 2, $\bar{H}^k(M(f),X)$ is trivial for $k\leqslant n$ and free Abelian for $k=n+1$. Using the above exact sequence we have that $\bar{H}_k(M(f),X)=0$ for $k\leqslant n$ which concludes the proof. □

Our next goal is to shed some light on the kernel of $\bar{H}_n(f)$: $\bar{H}_n(X;\mathbb{Z})\to\bar{H}_n(Y;\mathbb{Z})$.

Theorem 2. *Let $f:X\to Y$ be a surjective map between compact metrizable spaces. Assume that there is $n\geqslant 0$ such that $\tilde{\bar{H}}_k(f^{-1}(y);\mathbb{Z})=0$ for all $y\in Y$ and for $k<n$. If Y is the union of closed subsets $B_1,\ldots,B_m$, then the kernel of*

$$\bar{H}_n(f):\ \bar{H}_n(X;\mathbb{Z})\to\bar{H}_n(Y;\mathbb{Z})$$

is contained in the subgroup of $\tilde{\bar{H}}_n(X;\mathbb{Z})$ generated by the images of $\tilde{\bar{H}}_n(f^{-1}(B_i)$: $\mathbb{Z})\to\tilde{\bar{H}}_n(X;\mathbb{Z})$ for $i\leqslant m$.

Proof. We are going to prove Theorem 2 by induction on m. For $m=1$ it is obvious. Suppose Theorem 2 is true for $m<l$ and consider the case $m=l$. Let $A_m=f^{-1}(B_m)$, $B=\bigcup_{i=1}^{m-1}B_i$, $A=f^{-1}(B)$. Let Z be the mapping cylinder of f, $p:Z\to Y$ the projection

and $C = p^{-1}(B)$, $C_m = p^{-1}(B_m)$. Notice that the kernel of $\bar{H}_n(f)$ is the image of ∂: $\bar{H}_{n+1}(Z, X) \to \bar{H}_n(X)$. Consider Mayer-Vietoris sequence (see [11, p. 275])

$$\bar{H}_{n+1}(C_m, A_m) \oplus \bar{H}_{n+1}(C, A) \to \bar{H}_{n+1}(Z, X) \to \bar{H}_n(C_m \cap C, A_m \cap A) = 0.$$

Then for $\alpha \in \bar{H}_{n+1}(Z, X)$ there exist $\beta \in \bar{H}_{n+1}(C_m, A_m)$, $\gamma \in \bar{H}_{n+1}(C, A)$ with $\alpha = \beta + \gamma$. Take $\alpha' = \partial\alpha \in \bar{H}_n(X)$, $\beta' = \partial\beta \in \bar{H}_n(A_m)$, $\gamma' = \partial\gamma \in \bar{H}_n(A)$. Since $f_*(\alpha') = 0$ we have $f_*(\beta' + \gamma') = 0$. Therefore, $\beta' + \gamma' = 0$ in $\bar{H}_n(Z)$ and another Mayer-Vietoris sequence

$$\bar{H}_n(C \cap C_m) \to \bar{H}_n(C) \oplus H_n(C_m) \to \bar{H}_n(Z)$$

gives us the existence of $\eta \in \bar{H}_n(C \cap C_m)$ such that $\eta = \beta'$ in $\bar{H}_n(C_m)$ and $\eta = -\gamma'$ in $\bar{H}_n(C)$. By the Dual to Vietoris-Begle Theorem we may assume that $\eta \in \bar{H}_n(A \cap A_m)$. Then $\eta + \gamma' = 0$ in $H_n(C)$ and by the inductive assumption $\eta + \gamma'$ belongs to the subgroup of $\bar{H}_n(A)$ generated by the images of

$$\bar{H}_n(f^{-1}(B_k)) \to \bar{H}_n(A)$$

for $k \leq m - 1$.

Now $\alpha' = \beta' + \gamma' = (\beta' - \eta) + (\eta + \gamma')$ belongs to the subgroup of $\bar{H}_n(X)$ generated by the images of

$$\bar{H}_n(f^{-1}(B_k)) \to \bar{H}_n(X)$$

for $k \leq m$ because $\beta' - \eta \in \bar{H}_n(f^{-1}(B_m))$. □

Corollary 3. *Let f: $X \to Y$ be a surjective map between compact metrizable spaces. Assume that there is $n \geq 0$ such that $\bar{H}_k(f^{-1}(y); \mathbb{Z}) = 0$ for all $y \in Y$ and $k < n$. If each $y \in Y$ has a closed neighborhood V such that the inclusion induced homomorphism*

$$\bar{H}_n(f^{-1}(V); \mathbb{Z}) \to \bar{H}_n(X; \mathbb{Z})$$

is trivial then $\bar{H}_n(f)$: $\bar{H}_n(X; \mathbb{Z}) \to \bar{H}_n(Y; \mathbb{Z})$ is an isomorphism.

Remark. The reader should compare Theorem 2 and Corollary 3 with results of Soloway (see [9, 14]) for uv^{n-1} maps and singular homology.

To derive further corollaries we need some preliminary results.

Let X be a compact metrizable space. If U is a finite covering of X and π is a partition of unity subordinated to U then we have a map π: $X \to N(U)$, where $N(U)$ is the nerve of U. This gives rise to a homomorphism $\phi_U: \bar{H}_k(X; \mathbb{Z}) \to \bar{H}_k(N(U); \mathbb{Z})$ and by passing to the inverse limit we have a homomorphism

$$\phi_k: \ \bar{H}_k(X; \mathbb{Z}) \to \operatorname{inv\,lim} \bar{H}_k(N(U); \mathbb{Z}),$$

where the range is simply the kth Čech homology group $\check{H}_k(X; \mathbb{Z})$ of X with integer coefficients.

Obviously ϕ_k is a natural transformation of functors $\bar{H}_k$ and $\check{H}_k$.

Lemma 1. *ϕ_k is always an epimorphism. It is an isomorphism if and only if pro-$H_{k+1}(X;\mathbb{Z})$ satisfies the Miffag-Leffler condition. If $\bar{H}_k(X;\mathbb{Z})$ is countable then ϕ_k is an isomorphism.*

Proof. Let $X = \text{inv lim } X_n$, where each X_n is a compact polyhedron. Then we have the following exact sequence (see [5, p. 211]).

$$0 \to \lim^1 \bar{H}_{k+1}(X_n) \to \bar{H}_k(X) \to \text{inv lim } \bar{H}_k(X_n) \to 0$$

Since $\text{inv lim } \bar{H}_k(X_n) = \check{H}_k(X)$, ϕ_k is always an epimorphism. It is an isomorphism if and only if $\lim^1 \bar{H}_{k+1}(X_n) = 0$ which is equivalent (see [7]) to the fact that pro-$H_{k+1}(X) = \{\bar{H}_{k+1}(X_n)\}$ satisfies the Miffag-Leffler condition. If $\bar{H}_k(X)$ is countable, then $\lim^1 \bar{H}_{k+1}(X_n)$ is countable and [7] implies it has to be trivial. □

Lemma 2. *If ϕ_k: $\bar{H}_k(X;\mathbb{Z}) \to \check{H}_k(X;\mathbb{Z})$ is an isomorphism and $X = \text{inv lim } X_n$, where X_n are compact and metrizable, then the homomorphism*

$$\bar{H}_k(X;\mathbb{Z}) \to \text{inv lim } \bar{H}_k(X_n;\mathbb{Z})$$

is an isomorphism.

Proof. Since the composition of $\bar{H}_k(X) \to \text{inv lim } \bar{H}_k(X_n) \to \text{inv lim } \check{H}_k(X_n) = \check{H}_k(X)$ is an isomorphism, we infer that $\bar{H}_k(X) \to \text{inv lim } \bar{H}_k(X_n)$ is a monomorphism. On the other hand the exact sequence $0 \to \lim^1 \bar{H}_{k+1}(X_n) \to \bar{H}_k(X) \to \text{inv lim } \bar{H}_k(X_n) \to 0$ (see [5, p. 211] implies it is an epimorphism. □

Lemma 3. *Suppose A is a closed subset of a compact metrizable space X such that $\bar{H}_k(X, A;\mathbb{Z})$ is countable for some $k \geq 0$. Then there is a closed neighborhood V of A in X such that $\tilde{\bar{H}}_k(V;\mathbb{Z}) \to \tilde{\bar{H}}_k(X;\mathbb{Z})$ has the same image as $\tilde{\bar{H}}_k(A;\mathbb{Z}) \to \tilde{\bar{H}}_k(X;\mathbb{Z})$.*

Proof. First consider the case where $A = \{x\}$ is a one-point space. Take a decreasing sequence $V_1 \supset V_2 \cdots$ of closed neighborhoods of x in X such that $\bigcap_{m=1}^{\infty} V_m = \{x\}$. From the exact sequence (see [5, p. 211]) $0 \to \lim^1 \tilde{\bar{H}}_{q+1}(V_m) \to \tilde{\bar{H}}_q(x) \to \text{inv lim } \tilde{\bar{H}}_q(V_m) \to 0$ we infer that

$$\lim^1 \tilde{\bar{H}}_q(V_m) = \text{inv lim } \tilde{\bar{H}}_q(V_m) = 0$$

for all q. Let $G_m = \text{im}(\tilde{\bar{H}}_k(V_m) \to \tilde{\bar{H}}_k(X))$ and $F_m = \ker(\tilde{\bar{H}}_k(V_m) \to \tilde{\bar{H}}_k(X))$ for each m. Then the inverse sequence of short exact sequences

$$0 \to F_m \to \tilde{\bar{H}}_k(V_m) \to G_m \to 0$$

gives rise to the following exact sequence (see [11, p. 401]) $0 \to \text{inv lim } F_m \to \text{inv lim } \tilde{\bar{H}}_k(V_m) \to \text{inv lim } G_m \to \lim^1 F_m \to \lim^1 \tilde{\bar{H}}_k(V_m) \to \lim^1 G_m \to 0$.

Hence $\lim^1 G_m = 0$ and since each G_m is countable, we conclude from [7] that $\{G_m\}$ satisfies the Miffag-Leffler condition. In our case (G_m is a decreasing sequence of groups) it means that there is $N \geq 1$ such that $G_m = G_{m+1}$ for $m \geq N$.

From the exact sequence

$$\tilde{\tilde{H}}_k(V_m) \to \tilde{\tilde{H}}_k(X) \to \tilde{\tilde{H}}_k(X, V_m) = \tilde{\tilde{H}}_k(X/V_m)$$

we get that G_N is in the kernel of

$$\tilde{\tilde{H}}_k(X) \to \tilde{\tilde{H}}_k(X/V_m)$$

for all $m \geq N$. Therefore G_N is in the kernel of $\tilde{\tilde{H}}_k(X) \to \text{inv}\lim \tilde{\tilde{H}}_k(X/V_m)$ and since $X = \text{inv}\lim(X/V_m)$, Lemma 2 implies $G_N = 0$. Thus $V = V_N$ satisfies the needed condition.

Now consider the general case. Then $\tilde{\tilde{H}}_k(X/A) = \tilde{\tilde{H}}_k(X, A)$ is countable and there exists a closed neighborhood V of A such that

$$\tilde{\tilde{H}}_k(V/A) \to \tilde{\tilde{H}}_k(X/A)$$

is trivial. Each row in the diagram

$$\begin{array}{ccccc} \tilde{\tilde{H}}_k(A) & \longrightarrow & \tilde{\tilde{H}}_k(V) & \longrightarrow & \tilde{\tilde{H}}_k(V/A) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\tilde{H}}_k(A) & \longrightarrow & \tilde{\tilde{H}}_k(X) & \longrightarrow & \tilde{\tilde{H}}_k(X/A) \end{array}$$

is exact and therefore $\text{im}(\tilde{\tilde{H}}_k(V) \to \tilde{\tilde{H}}_k(X)) \subset \text{im}(\tilde{\tilde{H}}_k(A) \to \tilde{\tilde{H}}_k(X))$ which concludes the proof of Lemma 3. □

Corollary 4. *Let $f: X \to Y$ be a surjective map between compact metric spaces. Assume that there is $n \geq 0$ such that $\tilde{\tilde{H}}_k(f^{-1}(y); \mathbb{Z}) = 0$ for all $y \in Y$ and for $k < n$. If $\tilde{\tilde{H}}_n(Y; \mathbb{Z})$ is countable, then there exists $\varepsilon > 0$ such that for each finite closed covering $\{B_i\}_{i=1}^m$ of Y with sets of diameter less than ε, the kernel of*

$$\tilde{\tilde{H}}_n(f) \colon\ \tilde{\tilde{H}}_n(X; \mathbb{Z}) \to \tilde{\tilde{H}}_n(Y; \mathbb{Z})$$

is the subgroup of $\tilde{\tilde{H}}_n(X; \mathbb{Z})$ generated by the images of $\tilde{\tilde{H}}_n(f^{-1}(B_i); \mathbb{Z}) \to \tilde{\tilde{H}}_n(X; \mathbb{Z})$ for $i \leq m$.

Proof. By Lemma 3 for each $y \in Y$ there exists a closed neighborhood V_y of y such that $\tilde{\tilde{H}}_k(V_y) \to \tilde{\tilde{H}}_k(Y)$ is trivial. Let ε be the Lebesque number of the open covering $\{\text{Int } V_y\}_{y \in Y}$. If B is a closed subset of Y with diam $B < \varepsilon$, then $B \subset V_y$ for some $y \in Y$ and the homomorphism $\tilde{\tilde{H}}_n(f^{-1}(B)) \to \tilde{\tilde{H}}_n(Y)$ is trivial. Therefore Theorem 2 implies Corollary 4. □

Corollary 5. *Let $f: X \to Y$ be a surjective map between compact metrizable spaces. Assume that there is $n > 0$ such that $\tilde{\tilde{H}}_k(f^{-1}(y); \mathbb{Z}) = 0$ for all $y \in Y$ and $k < n$. If $\tilde{\tilde{H}}_n(X; \mathbb{Z})$ is countable, then there exists a finite subset S of Y such that the kernel of*

$$\tilde{\tilde{H}}_n(f) \colon\ \tilde{\tilde{H}}_n(X; \mathbb{Z}) \to \tilde{\tilde{H}}_n(Y; \mathbb{Z})$$

is the subgroup of $\tilde{\tilde{H}}_n(X; \mathbb{Z})$ generated by the images of $\tilde{\tilde{H}}_n(f^{-1}(y); \mathbb{Z}) \to \tilde{\tilde{H}}_n(X; \mathbb{Z})$ for $y \in S$.

Proof. From the exact sequence $\check{\tilde{H}}_n(X) \to \check{\tilde{H}}_n(X, f^{-1}(y)) \to \check{\tilde{H}}_{n-1}(f^{-1}(y)) = 0$ we get that $\check{\tilde{H}}_n(X, f^{-1}(y))$ is countable for each $y \in Y$.

By Lemma 3 for each $y \in Y$ there exists a closed neighborhood V_y of y in Y such that

$$\operatorname{im}(\check{\tilde{H}}_n(f^{-1}(V_y)) \to \check{\tilde{H}}_n(X)) = \operatorname{im}(\check{\tilde{H}}_n(f^{-1}(y)) \to \check{\tilde{H}}_n(X)).$$

Choose a finite set $S = \{y_1, \ldots, y_p\}$ such that $B_i = V_{y_i}$ is a covering of Y. Now Corollary 5 follows from Theorem 2. □

Example. Let X be the union of $[0,1] \times \{0\} \times \{0\}$ and the 2-spheres S_n of the radius 2^{-n-2} and the center $(2^{-n}, 2^{-n-2}, 0)$, $n = 0, 1, \ldots$. The map $f: X \to Y = [0,1] \times \{0\} \times \{0\}$ is the retraction such that $f(S_n) = \{(2^{-n}, 0, 0)\}$ for $n \geq 0$. Then $\check{\tilde{H}}_1(f^{-1}(y), \mathbb{Z}) = 0$ for $y \in Y$ and the kernel of $\bar{H}_2(f)$ equals $\bar{H}_2(X) = \check{H}_2(X)$ which is uncountable. Therefore, the kernel of $\bar{H}_2(f)$ cannot be generated by the images of

$$\bar{H}_2(f^{-1}(y); \mathbb{Z}) \to \bar{H}_2(X; \mathbb{Z}).$$

3. Vietoris–Begle theorem for pro-homology

Theorem 3. *Let R be a PID. Suppose $f: X \to Y$ is a surjective map of compact metrizable spaces such that pro-$\tilde{H}_k(f^{-1}(y); R)$ is trivial for all $y \in Y$ and for $k < n$. Then*

(a) *pro-$H_k(f)$: pro-$H_k(X; R) \to$ pro-$H_k(Y; R)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

(b) *If pro-$\tilde{H}_n(f^{-1}(y); R) \to$ pro-$\tilde{H}_n(X; R)$ is trivial for all $y \in Y$, then pro-$H_n(f)$ is an isomorphism.*

(c) *If pro-$\tilde{H}_n(f^{-1}(y); R)$ is semistable for each $y \in Y$, then $\check{H}_n(f)$: $\check{H}_n(X; R) \to \check{H}_n(Y; R)$ is an epimorphism. If the kernel of $\check{H}_n(f)$ is countable, then it is generated by the images of $\check{\tilde{H}}_n(f^{-1}(y); R) \to \check{\tilde{H}}_n(X; R)$. If pro-$H_n(X; R)$ is stable, then pro-$H_n(Y, R)$ is stable and the kernel of $\check{H}_n(f)$ is generated by the images of $\check{\tilde{H}}_n(f^{-1}(y); R) \to \check{\tilde{H}}_n(X; R)$ for $y \in Y$.*

Before proceeding with a proof of Theorem 3 we need the following:

Lemma 4. *Let R be a PID. If (B, A) is a pair of compact metrizable spaces, then the following conditions are equivalent $(n \geq 0)$:*

(a) *pro-$\tilde{H}_k(B, A; R)$ is trivial for $k \leq n$,*

(b) *$\check{\tilde{H}}^k(B, A; R)$ is trivial for $k \leq n$ and torsionfree for $k = n+1$.*

Proof. Because $H_k(P, Q) = \tilde{H}_k(P/Q)$ and $H^k(P, Q) = \tilde{H}^k(P/Q)$ for all polyhedral pairs (P, Q) it suffices to prove Lemma 4 for the case when $A = $ point.

Let $(B, A) = \operatorname{inv}\lim P_i$, where P_i are pointed polyhedra. If (a) is satisfied, then the direct sequence

$$0 \to \operatorname{Ext}(H_{k-1}(P_i; R), R) \to H^k(P_i; R) \to \operatorname{Hom}(H_k(P_i; R), R) \to 0$$

of short exact sequences (see [15, p. 243]) gives $H^k(B, A; R) = \operatorname{dir\,lim} \operatorname{Hom}(H_k(P_i; R), R)$ for $k \leq n$. Thus (b) follows.

If (b) is satisfied, then the inverse sequence

$$0 \to \operatorname{Ext}(H^{k+1}(P_i; R), R) \to H_k(P_i; R) \to \operatorname{Hom}(H^k(P_i; R), R) \to 0$$

of short exact sequences (see [15, p. 248]) implies that pro-$H_k(B, A; R)$ is trivial for $k \leq n$.

Indeed, for each i there exists $j > i$ such that the homomorphism $H^k(P_i, R) \to H^k(P_j; R)$ annihilates the torsion of $H^k(P_i; R)$ (is trivial) for $k \leq n+1$ (for $k \leq n$). Therefore, $H^k(P_i; R) \to H^k(P_j; R)$ factors through a free R-module and

$$\operatorname{Ext}(H^k(P_j; R), R) \to \operatorname{Ext}(H^k(P_i; R), R)$$

is trivial for $k \leq n+1$. Similarly

$$\operatorname{Hom}(H^k(P_j; R), R) \to \operatorname{Hom}(H^k(P_i; R), R)$$

is trivial for $k \leq n$. This concludes the proof of Lemma 4. □

Proof of Theorem 3. By Lemma 4, $\tilde{H}^k(f^{-1}(y); R)$ is trivial for $k < n$ and is torsion free for $k = n$. By Theorem 1 and Corollary 2, $\tilde{H}^k(M(f), X; R)$ is trivial for $k \leq n$ and torsion free for $k = n+1$. By Lemma 4, pro-$H_k(M(f), X; R)$ is trivial for $k \leq n$ and (a) follows.

Embed X in the Hilbert cube Q and let $F\colon Q \to Z$ be a map such that Y is a closed subset of $Z = F(Q)$, $F/X = f$, $X = F^{-1}(Y)$ and $F/Q - X$ is one-to-one. Choose a decreasing sequence $X_1 \supset X_2 \cdots$ of ANR's in Q with $X = \bigcap_{p=1}^{\infty} X_p$. Let $Y_p = f(X_p)$ and $f_p = F/X_p\colon X_p \to Y_p$ for $p \geq 1$.

Suppose pro-$\tilde{H}_n(f^{-1}(y); R) \to$ pro-$\tilde{H}_n(X; R)$ is trivial for all $y \in Y$ and fix $p \geq 1$. Then, for each $y \in Y_p$ there exists a compact neighborhood V_y of y in Y_p such that $\tilde{H}_k(f^{-1}(V_y); R) \to \tilde{H}_k(X_p; R)$ is trivial. Choose a finite subset $\{y_1, \ldots, y_m\}$ of Y_p such that $Y_p = \bigcup_{i=1}^m V_i$, where $V_i = V_{y_i}$, $i \leq m$.

Our goal is to show that pro-$H_n(f_p)$: pro-$H_n(X_p; R) \to$ pro-$H_n(Y_p; R)$ is an isomorphism. This follows from the following:

Special case of (b). Suppose that for a compact subset B and Y and $A = f^{-1}(B)$, $f/X - A$ is one-to-one and pro-$\tilde{H}_n(A; R) \to$ pro-$\tilde{H}_n(X; R)$ is trivial. Then pro-$H_n(f)$ is an isomorphism.

Proof of the special case. Consider the diagram

$$\begin{array}{ccccc} \text{pro-}\tilde{H}_n(A; R) & \longrightarrow & \text{pro-}\tilde{H}_n(X; R) & \longrightarrow & \text{pro-}H_n(X, A; R) \\ \downarrow & & \downarrow & & \downarrow \\ \text{pro-}\tilde{H}_n(B; R) & \longrightarrow & \text{pro-}\tilde{H}_n(Y; R) & \longrightarrow & \text{pro-}H_n(Y, B; R). \end{array}$$

By the part (a) of Theorem 3, the first two vertical homomorphisms are epimorphisms. Hence pro-$\tilde{H}_n(B; R) \to$ pro-$\tilde{H}_n(Y; R)$ is trivial. Also $X/A = Y/B$

implies that

$$\text{pro-}H_n(X, A; R) \to \text{pro-}H_n(Y, B; R)$$

is an isomorphism. Thus pro-$H_n(f)$ is an isomorphism.

Now (b) follows by applying its special case m times (first step is to consider the decomposition of X_p whose fibers are those of f_p which are contained in V_1, and one-point sets) and then observing that pro-$H_n(X; R)$ (pro-$H_n(Y; R)$) is the inverse limit of pro-$H_n(X_p; R)$ (pro-$H_n(Y_p; R)$) in the category of pro-(R-modules) (see [2, Lemma 3.4]).

So it remains to prove part (c) of Theorem 3.

Special case of part (c). Assume pro-$H_n(X; R)$ is stable and $A \subset X$ is the only non-trivial fiber of f.

Proof of the special case. Here $Y = X/A$ and pro-$\tilde{H}_n(Y; R)$ is isomorphic to pro-$H_n(X, A; R)$. Consider the exact sequence

$$\text{pro-}\tilde{H}_n(A; R) \to \text{pro-}\tilde{H}_n(X; R) \to \text{pro-}\tilde{H}_n(Y; R) \to 0.$$

Since pro-$H_n(f)$ is an epimorphism, pro-$\tilde{H}_n(Y; R)$ is Mittag-Leffler (semistable in today's terminology). As in Lemma 2.2 of [3] one can show that if in the exact sequence $G_4 \to G_3 \to G_2 \to G_1$ of pro-groups, G_i is stable for $i = 1, 3$ and semistable for $i = 2, 4$, then G_2 is stable.

Thus pro-$\tilde{H}_n(Y; R)$ is stable and by passing to the inverse limit we still have an exact sequence

$$\check{\tilde{H}}_n(A; R) \to \check{\tilde{H}}_n(X; R) \to \check{\tilde{H}}_n(Y; R) \to 0$$

(see [11, p. 401]) which is exactly what we need in the special case of (c).

Fix $p \geqslant 1$. For each $y \in Y_p$ there exists a compact neighborhood V_y of y in Y_p such that

$$\text{im}(\check{\tilde{H}}_n(f^{-1}(y); R) \to \tilde{H}_n(X_p; R)) = \text{im}(\tilde{H}_n(W_y; R) \to \tilde{H}_n(X_p; R))$$

for some $W_y \in \text{ANR}$ containing $f^{-1}(V_y)$ (this is because pro-$H_n(f^{-1}(y); R)$ is semistable). Choose a finite subset $\{y_1, \ldots, y_m\}$ of Y_p such that $y_i \neq y_j$ for $i \neq j$ and $Y_p = \bigcup_{i=1}^m V_i$, where $V_i = V_{y_i}$, $i \leqslant m$. Let Z_p be obtained from X_p by contracting each $f^{-1}(y_i)$, $i \leqslant r$, to a point. By the special case of (c) applied n times, pro-$H_n(Z_p; R)$ is stable and $\check{H}_n(X_p; R) \to \check{H}_n(Z_p; R)$ is an epimorphism whose kernel is generated by images of $\check{\tilde{H}}_n(f^{-1}(y_i); R) \to \check{\tilde{H}}_n(X_p; R)$, $i \leqslant m$. Let $g: Z_p \to Y_p$ be the map induced by f_p. Then for any $y \in Y_p$ there exists $i \leqslant m$ such that $f^{-1}(y) \subset f^{-1}(V_i)$ and since

$$\text{pro-}\tilde{H}_n(f^{-1}(V_y); R) \to \text{pro-}\tilde{H}_n(Z_p; R)$$

is trivial, pro-$\tilde{H}_n(g^{-1}(y); R) \to$ pro-$\tilde{H}_n(Z_p; R)$ is trivial. By part (b) of Theorem 3,

$$\text{pro-}H_n(Z_p; R) \to \text{pro-}H_n(Y_p; R)$$

is an isomorphism and therefore

$$\check{H}_n(Z_p; R) \to \check{H}_n(Y_p; R)$$

is an isomorphism. Thus, for each p,

$$\check{H}_n(X_p;\, R) \to \check{H}_n(Y_p;\, R)$$

is an epimorphism whose kernel F_p is generated by the images $\check{\check{H}}_n(F^{-1}(y);\, R) \to \check{\check{H}}_n(X_p;\, R),\ y \in Y_p$.

Take the inverse sequence of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{p+1} & \longrightarrow & \check{H}_n(X_{p+1};\, R) & \longrightarrow & \check{H}_n(Y_{p+1};\, R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_p & \longrightarrow & \check{H}_n(X_p;\, R) & \longrightarrow & \check{H}_n(Y_p;\, R) & \longrightarrow & 0. \end{array} \qquad (*)$$

Observe that $F_{p+1} \to F_p$ is an epimorphism (it follows from the description of F_p). By passing to the inverse limit we get the following exact sequence (see [11, p. 401])

$$0 \to \operatorname{inv\,lim} F_p \to \check{H}_n(X;\, R) \to \check{H}_n(Y;\, R) \to \lim{}^1 F_p.$$

In our case $\lim^1 F_p = 0$ because $\{F_p\}$ is Mittag-Leffler. Therefore $\check{H}_n(f)$: $\check{H}_n(X;\, R) \to \check{H}_n(Y;\, R)$ is epi. If pro-$H_n(X;\, R)$ is stable, then as before $(*)$ implies that $\{\check{H}_n(Y_p;\, R)\}$ is stable (see Lemma 2.2 in [3]) and Lemma 3.4 in [2] says that $\{\check{H}_n(Y_p;\, R)\}$ is isomorphic to pro-$H_n(Y;\, R)$ (it is because pro-$H_n(Y_p;\, R)$ is stable for all $p \geq 1$). Using Lemma 2.2 of [3] once again we get that $\{F_p\}$ is stable if pro-$H_n(X;\, R)$ is stable. We reach the same conclusion if the kernel of $\check{H}_n(f)$ is countable: that kernel is isomorphic to inv lim F_p and Corollary 6.1.9. of [4, p. 81] says that $\{F_p\}$ is stable. In our case it means that $F_{p+1} \to F_p$ is an isomorphism for sufficiently large p. We assume it is so for all p.

Let F be the subgroup of $\check{\check{H}}_n(X;\, R)$ generated by the images of $\check{\check{H}}_n(f^{-1}(y);\, R) \to \check{\check{H}}_n(X;\, R)$ for $y \in Y$. The homomorphism

$$\check{\check{H}}_n(X;\, R) \to \check{\check{H}}_n(X_p;\, R)$$

sends F onto F_p. Since $\check{\check{H}}_n(X;\, R) = \operatorname{inv\,lim} \check{\check{H}}_n(X_p;\, R)$, $F \to \operatorname{inv\,lim} F_p$ is a monomorphism. Therefore $F \to F_p$ is a monomorphism for each p and, consequently, $F = \operatorname{inv\,lim} F_p$. □

Remark. The reader should compare Theorem 3 with results of Soloway (see [9, 14]) for uv^{n-1} maps and the singular homology.

References

[1] G. Bredon, Sheaf Theory (McGraw-Hill, New York, 1967).

[2] J. Dydak, On LC^n-divisors, Topology Proceedings 3 (1978) 319–333.

[3] J. Dydak, Local n-connectivity of quotient spaces and one-point compactifications, Shape Theory and Geom. Top. Proc., Dubrovnik, 1981, Lecture Notes in Math. 870 (Springer, Berlin, 1981) 48–72.

[4] J. Dydak and J. Segal, Shape theory: An introduction, Lecture Notes in Math. 688 (Springer, Berlin, 1978) 1–150.

[5] D.A. Edwards and H.M. Hastings, Čech and Steenrod Homotopy Theories With Applications to Geometric Topology, Lecture Notes in Math. 542 (Springer, Berlin, 1976).
[6] L. Fuchs, Infinite Abelian Groups Vol. I (Academic Press, New York/London, 1970).
[7] B.I. Gray, Spaces of the same n-type for all n, Topology 5 (1966) 241-243.
[8] P.J. Hilton and U. Stammbach, A Course in Homology Algebra, (Springer, Berlin).
[9] R.C. Lacher, Cell-like mappings and their generalizations, Bull. Amer. Math. Soc. 83 (1977) 495-552.
[10] J.D. Lawson, Comparison of taut cohomologies, Aequationes Math. 9 (1973) 201-209.
[11] W.S. Massey, Homology and Cohomology Theory, (Dekker, New York, 1978).
[12] E. Michael, Local properties of topological spaces, Duke Math. J. 21 (1954) 163-171.
[13] J. Milnor, On axiomatic homology theory, Pac. J. Math. 12 (1962) 337-341.
[14] R.N. Soloway, Somewhere acyclic mappings of manifolds are compact, Ph.D. thesis, University of Wisconsin, 1971.
[15] E.H. Spanier, Algebraic Topology (McGraw-Hill, New York, 1966).
[16] A.Ju. Volovikov and Nguen Le Anh, On the Vietoris-Begle Theorem, Vestnik Moskov. Univ. Ser. I Math. Mekh. 1984 no. 3, 70-71.