Proof. Exactly as above, $2^{\varkappa}$ collapses to cf $\varkappa$ in the extension. In this case we can only say that the union of a set of M of compatible conditions of cardinality less than cf $\varkappa$ is again a condition; so cardinals less than or equal to cf $\varkappa$ will be preserved.

Note. In this case, the set of conditions P will have cardinality in M, $\varkappa^{\varkappa}$; this may be greater than $2^{\varkappa}$, and we do not know whether this also collapses to cf $\varkappa$.

COROLLARY 4. *If the real cardinal of $2^{\varkappa}$ of M is greater than the real cardinal of* cf $\varkappa$ *of M, then there is no set generic over M for this notion of forcing.*

Proof. By Corollaries 2 and 3.

In the case of $\varkappa$ singular, another notion of forcing is immediately suggested, which turns out to be simpler to deal with than the notion above: namely, to take as conditions, those partial functions in M from $\varkappa$ into $\{0,1\}$, whose domain is bounded by an ordinal less than $\varkappa$. (Clearly this coincides with the previous notion for regular $\varkappa$.) Assuming G' is generic over M for this second notion, we can prove:

THEOREM 5. *If a is an ordinal with* cf $\varkappa < a \leqslant 2^{\varkappa}$, *then in the extension $M[G']$, a is similar to* cf $\varkappa$; *all cardinals outside this range are preserved.*

Proof. The proof that for a cardinal $a < \varkappa$ of M, 2^{a} collapses to cf $\varkappa$ in $M[G']$, can be taken over from Theorem 1 without change (though it is essentially simpler in this case); and the proof that cardinals less than or equal to cf $\varkappa$ are preserved is as in Corollary 3.

To see that cardinals greater than $2^{\varkappa}$ are now preserved we simply note that the set P' of conditions in the new sense has cardinality $2^{\varkappa}$ in M.

Added in proof: Since presenting this paper, the author has been informed, that some of these results were known previously: in particular the case $\varkappa = \aleph_1$ was known to Vopěnka. Also Jech has pointed out that the question noted after Corollary 3 can be answered negatively using results of Engelking and Karlowicz.

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Modified Vietoris theorems for homotopy

by

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1. Introduction. Smale's Vietoris theorem for homotopy [9] and its various generalizations ([5], [8]) impose local connectivity conditions on the fibres of the given map $p\colon X \to Y$; in this paper we obtain versions that depend on the manner that the fibers of p are embedded in X rather than on their actual structure.

In the first part (§ 2) we study a condition, called PC_X^n, on the embedding of a set A in a space X; in particular (2.4) suitable conditions on A itself are sufficient (but not necessary) for A to be PC_X^n. In § 3, 4, upper semi-continuous decompositions of a space X into PC_X^n subsets having a paracompact decomposition space Y are characterized; under certain assumptions (4.4–4.7), for example, when Y is metrizable, then Y must have strong local properties. The Vietoris-type theorems for $p\colon X \to Y$ are given in § 5; the general result (5.1) can be improved considerably if either Y is dominated by a polytope (5.2) or if Y has suitable local properties. Some applications are given in § 6.

2. Proximally n-connected sets. In writing homotopy groups, the base point will be omitted unless explicitly needed. Let $A \subset B$; for $n \geqslant 1$ we denote by $\pi_n(A|B)$ the image of $\pi_n(A)$ in $\pi_n(B)$ under the homomorphism induced by the inclusion map; $\pi_0(A|B) = 0$ will denote that any two points of A can be joined by a path in B.

2.1. DEFINITION. Let X be a Hausdorff space. The set $A \subset X$ is called *proximally n-connected in X* (written: n-PC_X) if for each neighborhood $U(A)$ of A in X there is a neighborhood $V(A) \subset U$ of A in X such that $\pi_n(V|U) = 0$. The set A is PC_X^n if it is k-PC_X for all $0 \leqslant k \leqslant n$; and A is PC_X^{∞} if it is PC_X^n for every $n \geqslant 0$.

This notion reduces to that of LC^n ([1], [2], [6]) whenever A is a single point, in that a_0 is PC_X^n if and only if X is LC^n at a_0. No 0-PC_X set can be embedded into two disjoint open subsets so, in particular, a closed 0-PC_X subset of a normal X is necessarily connected. Other than this, even the

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strong requirement that A be PC_X^∞ does not impose severe limitations (such as n-connectedness, or local connectedness) on the structure of A itself: in $X = E^3$, bend the tube $\{(x, y, z) | x^2+y^2 = (1/z)^4, z \geqslant 1\}$ to form a $\sin(1/t)$-shaped surface S that converges to the line segment $L = \{(x, 0, 0) | 3 \leqslant x \leqslant 4\}$ as $z \to \infty$; then the (closed) set $A = S \cup L$ is PC_X^∞.

The condition PC_X^n is therefore a condition on the embedding of A in X, rather than on the structure of A itself; and if A is the intersection of a descending sequence $\{A_i | i = 1, 2, \ldots\}$ of compact PC_X^n sets, then A itself is PC_X^n since any $U(A)$ contains almost all the A_i. Moreover,

2.2. THEOREM. *If A is PC_X^n, then*

$$\pi_q(U, A) \approx \pi_q(U) \oplus \pi_{q-1}(A), \qquad 2 \leqslant q \leqslant n$$

for every open $U \supset A$.

Proof. Given U, find $W(A) \subset U$ such that $\pi_q(W|U) = 0$ for $0 \leqslant q \leqslant n$. Since the inclusion i: $A \to U$ factors through W, we find $\pi_q(A|U) = 0$ for $0 \leqslant q \leqslant n$ so that the exact homotopy sequence of the pair (U, A) breaks up, for $1 \leqslant q \leqslant n$, into a succession of short exact sequences

$$0 \to \pi_q(U) \xrightarrow{j} \pi_q(U, A) \xrightarrow{\partial} \pi_{q-1}(A) \to 0$$

(∂ is the boundary homomorphism, j is that induced by inclusion). Similarly, starting with W, we get the above short exact sequences with W replacing U.

For any fixed $2 \leqslant q \leqslant n$, we therefore have a commutative diagam of short exact sequences

$$\begin{array}{ccccccccc} 0 \to & \pi_q(W) & \xrightarrow{j} & \pi_q(W, A) & \xrightarrow{\partial} & \pi_{q-1}(A) & \to 0 \\ & \downarrow \lambda & & \downarrow \mu & & \downarrow \mathrm{id} & \\ 0 \to & \pi_q(U) & \xrightarrow[\hat{j}]{} & \pi_q(U, A) & \xrightarrow[\hat{\partial}]{} & \pi_{q-1}(A) & \to 0 \end{array}$$

where λ, μ are induced by inclusion so that λ is the zero homomorphism because $\pi_q(W|U) = 0$. Define s: $\pi_{q-1}(A) \to \pi_q(U, A)$ by setting $s(a) = \mu\partial^{-1}(a)$ for each $a \in \pi_{q-1}(A)$. Each $s(a)$ is a unique element of $\pi_q(U, A)$: for, if $\partial\beta = \partial\beta' = a$, then $(\beta - \beta') = j(\gamma)$ for some $\gamma \in \pi_q(W)$ and therefore $\mu(\beta-\beta') = \hat{j}\lambda(\gamma) = 0$. Since s is clearly a homomorphism, and since $\hat{\partial}s = \mathrm{id}$, the bottom short exact sequence splits, and the proof is complete.

Under certain conditions on X, the PC_X^n property of A follows from a simple property of A; for example, it is easy to see ([1], p. 87, [2], p. 239) that if X is an ANR, and if $A \subset X$ is either a closed AR, or the intersection of a descending sequence of compact AR, then A is PC_X^∞. To establish a somewhat more general such condition, we need the

2.3. LEMMA. *Let X be a paracompact LC^n space, and let $A \subset X$ be a closed LC^k subspace. Then given any neighborhood $U(A)$ there exists a neighborhood $V(A) \subset U$ with the property: for each $0 \leqslant q \leqslant \min[n, k+1]$, every f: $S^q \to V$ is homotopic in U to a map of S^q into A.*

Proof. Since X is LC^n, the open covering $\mathfrak{U} = \{U, X - A\}$ of X has an open refinement $\{W\}$ such that any two $\{W\}$-close [1] maps f, g: $P \to X$ of any polytope P, $\dim P \leqslant n$, are $\mathfrak{U}$-homotopic [2]. Let $\{W'\}$ be an open star-refinement [3] of $\{W\}$; we can assume $\{W'\}$ is nbd-finite.

Since A is LC^k, it follows ([4], p. 179) that there is a nbd-finite open covering $\{W''\}$ of X such that

(a) $\{A \cap W''\}$ is a refinement of $\{A \cap W'\}$.

(b) Any partial realization [4] of any polytope K, $\dim K \leqslant k+1$, in $\{A \cap W''\}$ extends to a full realization [5] in $\{A \cap W'\}$.

Let $\{V'\}$ be an open star-refinement of the open covering $\{W' \cap W''\}$ of X, and define

$$V = \bigcup \{V' | A \cap V' \neq \emptyset\}.$$

Then $A \subset V$; and also $V \subset U$ since each $V' \subset$ some W, and if $A \cap V' \neq \emptyset$ then $A \cap W \neq \emptyset$ so that $W \subset U$.

Now let f: $S^q \to V$ be given, and subdivide S^q simplicially so fine that $f(\overline{\mathrm{St}\,p}) \subset$ some $V' = V'(p) \in \{V'\}$ for each closed vertex-star $\overline{\mathrm{St}\,p}$. For each vertex p, let $g(p)$ be any element of $A \cap V'(p)$; then g is a partial realization of S^q in $\{A \cap W''\}$: for if, $\overline{\sigma} = (p_0, \ldots, p_q)$ is any q-simplex of S^q, then

$$f(\overline{\sigma}) \subset \bigcap_0^q f(\mathrm{St}\,p_i) \subset \bigcap_0^q V'(p_i)$$

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(1) If X is any space, and $\mathfrak{U}$ any open covering, then two maps f, g: $R \to X$ of a space R into X are called $\mathfrak{U}$-close whenever $f(r)$ and $g(r)$ belong to a common $U \in \mathfrak{U}$ for each $r \in R$; f and g are $\mathfrak{U}$-homotopic if there is a homotopy H: $f \simeq g$ such that $H(r, I) \subset$ some $U \in \mathfrak{U}$ for each $r \in R$.

(2) A direct proof for paracompact spaces X is entirely analogous to that given for metric spaces X in ([2], p. 234) it is also a special case of Theorem 3.2 in the next section.

(3) A refinement $\mathfrak{U}^* = \{U^*\}$ of an open covering $\mathfrak{U}$ is called a star-refinement of $\mathfrak{U}$ if $\bigcup \{U^* | U^* \cap U_0^* \neq \emptyset\} \subset$ some $U \in \mathfrak{U}$ for each $U_0^* \in \mathfrak{U}^*$. By Stone's theorem ([4], p. 168) a space is paracompact if and only if each open covering has an open star-refinement.

(4) Let Y be any space, $\mathfrak{U}$ an open covering of Y, and P a polytope (not necessarily finite). A partial realization of P in $\mathfrak{U}$ is a (continuous) map f: $Q \to Y$ of some subpolytope $Q \subset P$ that contains the zero-skeleton P^0 of P, such that $f(Q \cap \overline{\sigma})$ is contained in some $U \in \mathfrak{U}$ for each closed simplex $\overline{\sigma}$ of P. The realization of P is called full if $Q = P$.

(5) The proof given in ([2], p. 234) is valid for paracompact spaces X; it is also a special case of Theorem 3.1 in the next section.

so, since $\{V'\}$ is a star-refinement,

$$f(\bar\sigma) \cup \bigcup_0^q g(p_i) \subset \bigcup_0^q V'(p_i) \subset \text{some } W_0' \cap W_0''$$

and therefore $\bigcup_0^q g(p_i) \subset A \cap W_0''$. Since $q \leqslant k+1$, we find g extends to a full realization of S^q in $\{A \cap W'\}$. Any W' containing $g(\bar\sigma)$ meets the W_0' above containing $f(\bar\sigma) \cup \bigcup_0^q g(p_i)$ so that $W' \cup W_0'$ (therefore also $f(\bar\sigma) \cup g(\bar\sigma)$) lies in a single $W \in \{W\}$ and, since $W \cap A \neq \emptyset$, we have $W \subset U$. Thus, f and g are $\mathfrak{U}$-homotopic, and the homotopy is actually over U. This completes the proof.

It now follows at once that

2.4. *Let X be a paracompact* LC^n *space and $A \subset X$ a closed* LC^k *subspace. If $\pi_q(A) = 0$ for some $0 \leqslant q \leqslant \min[n, k+1]$, then A is q-PC_X.*

As previously remarked, 2.4 remains true if A is the intersection of a descending sequence $\{A_i \mid i = 1, 2, \ldots\}$ of compact LC^k sets such that $\pi_q(A_i) = 0$ for all large i. In particular, if X is LC^n and if A is an n-connected closed LC^{n-1} subset (or the intersection of a descending sequence of such compact sets) then A is PC_X^n.

3. Decomposition spaces. In this section, we study upper semi-continuous decompositions of a space X into PC_X^n subsets or, equivalently, continuous closed surjections $p\colon X \to Y$ where each fiber $p^{-1}(y)$ is PC_X^n. We will show that such maps are characterized by a partial realization property (3.1) and also by a homotopy property (3.2); observe that by taking $p = \mathrm{id}$, these results give characterizations of LC^n in paracompact spaces analogous to those in [2], p. 234, [6], p. 265 for metric spaces.

If $p\colon X \to Y$ and $B \subset Y$, the set $p^{-1}(B) \subset X$ is denoted by $\tilde B$.

3.1. THEOREM. *Let X be arbitrary, Y paracompact, and $p\colon X \to Y$ a continuous closed surjection. The following two statements are equivalent:*

(a) *Each fiber of p is* PC_X^n.

(b) *For each open covering $\{U\}$ of Y there exists an open refinement $\{V\}$ with the property: Any partial realization of any polytope P, $\dim P \leqslant n+1$, in $\{\tilde V\}$ extends to a full realization in $\{\tilde U\}$.*

Proof. (a) $\Rightarrow$ (b). Denote the given covering $\{U\}$ by $\{U^{n+1}\}$ and for each $y \in Y$, let $U^{n+1}(y)$ be a definite set of the open covering $\{U^{n+1}\}$ that contains y. Construct a succesison of open coverings $\{U^s\}$, $s = n, n-1, \ldots, 0$, as follows:

$(n, 1)$ Let $W^{n+1}(\tilde y)$ be a neighborhood of $\tilde y = p^{-1}(y)$ such that

$$\pi_n\big(W^{n+1}(\tilde y) \mid \tilde U^{n+1}(y)\big) = 0\,.$$

$(n, 2)$ Let $G^{n+1}(y)$ be a nbd of y such that

$$p^{-1}(y) \subset p^{-1}\big(G^{n+1}(y)\big) \subset W^{n+1}(\tilde y)$$

(this exists since p is a closed map ([4] p. 86)).

$(n, 3)$ Let $\{U^n\}$ be an open star-refinement of

$$\{U^{n+1}(y) \cap G^{n+1}(y') \mid (y, y') \in Y \times Y\}\,.$$

We proceed recursively until $s = 0$: if $\{U^{s+1}\}$ is defined, repeat the above construction using $\{U^{s+1}\}$ and the s-PC_X property of the fibers to get $W^{s+1}(\tilde y)$, $G^{s+1}(y)$ and then $\{U^s\}$ as an open star-refinement of the open covering

$$\{U^{s+1}(y) \cap G^{s+1}(y') \mid (y, y') \in Y \times Y\}\,.$$

Each $\{U^s\}$ is clearly a refinement of $\{U^{s+1}\}$; we will show that the refinement $\{U^0\}$ of $\{U^{n+1}\}$ has the property stated in the theorem.

Let g be a partial realization of P in $\{\tilde U^0\}$; then $g\colon Q \cup P^0 \to X$ for some subpolytope $Q \subset P$ and $g\big(\bar\sigma \cap (Q \cup P^0)\big) \subset$ some $\tilde U^0$ for each closed simplex $\bar\sigma$ of P. We proceed by induction, assuming that for some $0 \leqslant r \leqslant n$, the map g has been extended to a partial realization $g^r\colon Q \cup P^r \to X$ of P in $\{\tilde U^r\}$.

Let $\bar\sigma^{r+1}$ be any fixed $(r+1)$-simplex of P; all the vertices of σ^{r+1} have images lying a single $\tilde U^0 \subset \tilde U_0^r$, and $g^r(\bar\sigma^r) \subset$ some $\tilde U^r = \tilde U^r(\bar\sigma^r)$ for each r-face $\bar\sigma^r$ of $\bar\sigma^{r+1}$. Thus $\tilde U_0^r \cap \tilde U^r(\bar\sigma^r) \neq \emptyset$ for each $\bar\sigma^r$ and, since $\{U^r\}$ is a star-refinement of $\{U^{r+1}(y) \cap G^{r+1}(y')\}$, this shows that $\bigcup\{g^r(\bar\sigma^r) \mid \bar\sigma^r$ a face of $\bar\sigma^{r+1}\} \subset$ some $\tilde G^{r+1}(y) \subset W^{r+1}(\tilde y)$; therefore $g^r \mid \dot\sigma^{r+1}$ is extendable to a $g^{r+1}\colon \bar\sigma^{r+1} \to \tilde U^{r+1}(y)$. Extending over each $(r+1)$-simplex in this manner extends the partial realization g^r to a partial realization $g^{r+1}\colon Q \cup P^{r+1} \to X$ of P in $\{\tilde U^{r+1}\}$, completing the inductive step, and the proof.

(b) $\Rightarrow$ (a). Given $y \in Y$, and any open $G \supset p^{-1}(y)$, choose a nbd $U(y)$ such that $\tilde y \subset \tilde U \subset G$ and then a nbd $W(y)$ such that $y \in W \subset \overline{W} \subset U$. Let $\mathfrak{U} = \{U, Y - \overline{W}\}$ and let $\{V\}$ be an open refinement satisfying (b). Choose any $V \in \{V\}$ containing y; then $V \subset U$. For any $0 \leqslant k \leqslant n$, each $f\colon S^k \to \tilde V$ is a partial realization of the ball H^{k+1} in $\tilde V$, hence extends to a full realization F of H^{k+1} in $\tilde{\mathfrak{U}}$ and, necessarily, $F(H^{k+1}) \subset \tilde U$. Thus, $p^{-1}(y)$ is PC_X^n and the proof is complete.

The companion characterization by homotopy is

3.2. THEOREM. *Let X be arbitrary, Y paracompact, and $p\colon X \to Y$ a continuous closed surjection. The following two statements are equivalent:*

(a) *Each fiber of p is* PC_X^n.

(b) *Each open covering $\{U\}$ of Y has an open refinement $\{W\}$ with the property: For any polytope P, $\dim P \leqslant n$, any two continuous $f, g\colon P \to X$ that are $\{\tilde{W}\}$-close are $\{\tilde{U}\}$-homotopic, and a homotopy can be chosen rel any subpolytope Q such that $f|Q = g|Q$.*

Proof. (a) $\Rightarrow$ (b). Let $\{U^*\}$ be a star-refinement of $\{U\}$, and let $\{W\}$ be an open refinement of $\{U^*\}$ satisfying 3.1(b). If $f, g\colon P \to X$ are $\{\tilde{W}\}$-close then $\{f^{-1}(\tilde{W}) \cap g^{-1}(\tilde{W})|\ W \in \{W\}\}$ is an open covering of P. Subdivide P simplicially so fine that each closed simplex lies in some set of this covering, and take $P \times I$ in the standard simplicial subdivision that introduces no new vertices other than those on $P \times 0$ and $P \times 1$. Let $L = (P \times 0) \cup (Q \times I) \cup P \times 1$ and define $H\colon L \to X$ by $H|P \times 0 = f$, $H|P \times 1 = g$, $H(q, t) = f(q) = g(q)$ for $(q, t) \in Q \times I$. Then H is a partial realization of $P \times I$ in $\{\tilde{W}\}$: for, any $(n+1)$-simplex $\bar{\sigma}$ of $P \times I$ is of the form $\bar{\sigma} = (p_0 \times 0, \ldots, p_i \times 0, p_i \times 1, \ldots, p_n \times 1)$, where $\tau = (p_0, \ldots, p_i, \ldots, p_n)$ is n-simplex of P so, because $f(\tau) \cup g(\tau) \subset$ some $\tilde{W}$ we find $H(\bar{\sigma} \cap L) \subset \tilde{W}$. Thus, H extends to a full realization of $P \times I$ in $\{\tilde{U}^*\}$, and this is easily seen to be a $\{\tilde{U}\}$-homotopy of f to g.

(b) $\Rightarrow$ (a). As in 3.1: given y and an open $U \ni y$, choose an open W such that $y \in W \subset \overline{W} \subset U$, and let $\{V\}$ be a refinement of the open covering $\{U, Y - \overline{W}\}$ satisfying (b). If $V \in \{V\}$ contains y, then any $f\colon S^k \to \tilde{V}$ $(0 \leqslant k \leqslant n)$ is $\{\tilde{V}\}$-close to the constant map of S^k to a point of $p^{-1}(y)$ so is null homotopic over $\tilde{U}$.

4. Characterization by function spaces. It is convenient to express the results 3.1, 3.2 in terms of function spaces.

The compact-open topology in Y^X will be called the c-topology. For each $f \in Y^X$ and each open covering $\mathfrak{U}$ of Y, let

$$(f, \mathfrak{U}) = \{g \in Y^X|\ g \text{ is } \mathfrak{U}\text{-close to } f\};$$

clearly $(f, \mathfrak{V}) \subset (f, \mathfrak{U})$ whenever $\mathfrak{V}$ refines $\mathfrak{U}$. We shall need the following useful (6)

4.1. Lemma. *Let X be compact. Then the family of all sets $\{(f, \mathfrak{U})\}$ forms a basis for the c-topology in Y^X.*

Proof. Let $(A, V) = \{f \in Y^X|\ f(A) \subset V\}$; the c-topology in Y^X has the family $\{(A, V)|\ A \text{ compact}, V \text{ open}\}$ as a sub-basis.

(i) Each $(f, \mathfrak{U})$ is open in the c-topology. Let $g \in (f, \mathfrak{U})$. For each $x \in X$ there is a $U(x) \in \mathfrak{U}$ such that $f(x) \cup g(x) \in U(x)$ so we can find a nbd $V(x)$ of x such that $\overline{V(x)}$ is compact and $f(\overline{V(x)}) \cup g(\overline{V(x)}) \subset U(x)$. Let $\{V(x_1), \ldots, V(x_n)\}$ be a finite subcovering for X; then $g \in G = \bigcap_1^n (\overline{V(x_i)}, U(x_i))$. Moreover, $G \subset (f, \mathfrak{U})$: for, let $h \in G$; given $x \in X$, we have $x \in V(x_i)$ for some i, so $h(x) \in U(x_i)$ and also $f(x) \in f(\overline{V(x_i)}) \subset U(x_i)$; thus, $h \in (f, \mathfrak{U})$.

(ii) The $\{(f, \mathfrak{U})\}$ form a basis. Let $f \in G = \bigcap_1^n (A_i, W_i)$, where G is a basic open set. For each $r = 1, \ldots, n$, let $\mathfrak{W}_r$ be the open covering $\{W_r, Y - f(A_r)\}$ of Y. Let $\mathfrak{U} = \{U_1 \cap \ldots \cap U_n|\ U_r \in \mathfrak{W}_r, 1 \leqslant r \leqslant n\}$; then $\mathfrak{U}$ is an open covering of Y, and we have $f \in (f, \mathfrak{U}) \subset G$: for, let $g \in (f, \mathfrak{U})$ and fix any A_i; for each $a \in A_i$ there must be a set of $\mathfrak{U}$ containing $g(a)$ and $f(a)$; but since $f(a) \notin Y - f(A_i)$ such a set must be from among those having W_i in the ith place, and all such sets are contained in W_i. Thus, $g(A_i) \subset W_i$ for each $i = 1, \ldots, n$ so $(f, \mathfrak{U}) \subset G$. This completes the proof (7).

Using the c-topology in the function spaces, recall that a continuous $p\colon X \to Y$ induces a continuous $p_\#\colon X^P \to Y^P$ by setting $p_\#(f) = p \circ f$, and that whenever P is (locally) compact, two maps $f, g\colon P \to X$ are homotopic if and only if they belong to the same path-component of X^P ([4], p. 320). With these preliminaries, a function-space formulation of 3.2 is

4.2. Theorem. *Let X be arbitrary, Y paracompact, and $p\colon X \to Y$ a continuous closed surjection. The following two statements are equivalent:*

(a) *Each fiber of p is PC_X^n.*

(b) *Let P be a finite polytope, $\dim P \leqslant n$, and let $f \in Y^P$. Given any nbd $(f, \mathfrak{U})$ of f, there exists a refinement $\mathfrak{V}^*$ of $\mathfrak{U}$ such that $p_\#^{-1}(f, \mathfrak{V}^*)$ is path-connected in $p_\#^{-1}(f, \mathfrak{U})$.*

Proof. (a) $\Rightarrow$ (b). Let $\mathfrak{U}^*$ be a star-refinement of $\mathfrak{U}$, let $\mathfrak{V}$ satisfy 3.2 relative to $\mathfrak{U}^*$, and let $\mathfrak{V}^*$ be a star-refinement of $\mathfrak{V}$. If $pg, pg' \in (f, \mathfrak{V}^*)$, then g, g' are $\tilde{\mathfrak{V}}$-close consequently there is a $\tilde{\mathfrak{U}}^*$-homotopy $H\colon g \simeq g'$; since $pH(x, I) \subset$ some U_0^*, $f(x) \cup pH(x, 0) \subset U_1^*$ and $f(x) \cup pH(x, 1) \subset U_2^*$, it follows that $pH(x, 1) \cup f(x) \subset$ some $U \in \mathfrak{U}$, consequently $p_\#^{-1}(f, \mathfrak{V}^*)$ is path-connected in $p_\#^{-1}(f, \mathfrak{U})$. (b) $\Rightarrow$ (a) is trivial.

For any $Q \subset P$ and any $g\colon Q \to X$, let $X^P(Q, g) \subset X^P$ be the (possibly empty) subspace of all extensions of g over P; if $Q = \emptyset$, this set is simply X^P. Theorem 3.1 implies a weak lifting property:

4.3. *Let X be arbitrary, Y paracompact, and $p\colon X \to Y$ a continuous closed surjection having each fiber PC_X^n. Then for any finite polytope P, $\dim P \leqslant n+1$, any subpolytope $Q \subset P$, and any $g\colon Q \to X$, the set $p_\#[X^P(Q, g)]$ is dense in $Y^P(Q, pg)$.*

(6) If Y is regular, X arbitrary, the topology in Y^X obtained by using the family $\{(f, \mathfrak{U})\}$ as sub-basis is easily seen to be admissible ([4], p. 274) so that it contains the c-topology.

(7) The proof shows slightly more: the family $\{(f, \mathfrak{U})|\ f \in Y^X, \mathfrak{U}$ a finite open covering of $Y\}$ forms a basis for the c-topology in Y^X whenever X is compact.

Proof. Let $G\colon P\to Y$ be any extension of pg; we are to show each $p_{\#}^{-1}(G,\mathfrak{U})$ contains an extension of g. Let $\mathfrak{U}^*$ be a star-refinement of $\mathfrak{U}$, and let $\mathfrak{V}$ satisfy 3.1 relative to $\mathfrak{U}^*$. Subdivide P so fine that $G(\bar\sigma)$ is contained in some $V\in\mathfrak{V}$ for each closed simplex $\bar\sigma$ of P. Define $g^0\colon Q\cup P^0\to X$ by $g^0|Q=g$ and $g^0(v)\in p^{-1}G(v)$ for each $v\in P^0-Q$. Then g^0 is a partial realization of P in $\widetilde{\mathfrak{V}}$, so it extends to a full realization $\widetilde{G}$ of P in $\widetilde{\mathfrak{U}}^*$, and $p\widetilde{G}\in(G,\mathfrak{U})$.

By imposing an additional condition on Y, these two results immediately give a necessary condition for the existence of surjections such as we are considering:

4.4. THEOREM. *Let X be arbitrary, Y paracompact, and $p\colon X\to Y$ a continuous closed surjection with PC_X^n fibers. If the space Y^{S^k} is first countable for some $0\leqslant k\leqslant n$, then Y^{S^k} is LC^0 and therefore Y is k-LC (and first countable).*

Proof. This will follow from the simple

4.5. LEMMA. *Let Z be a first countable space and let $D\subset Z$ be dense. Assume that for each $z\in Z$ and each* nbd $U(z)$ *there is a* nbd $V(z)$ *such that $V\cap D$ is path-connected in U. Then Z is LC^0.*

Proof of Lemma. We show that any two points of V can be joined by a path in U; for this, it suffices to show that each $v\in V$ can be joined to a point of $V\cap D$ by a path in U.

Let $U_1\supset U_2\supset\ldots$ be a countable basis at v. Proceeding inductively, define sets $V_1\supset V_2\supset\ldots$ with $v\in V_i\subset U_i$ as follows: find $V_1(v)\subset U\cap U_1$ such that $V_1\cap D$ is path connected in $U\cap U_1$; assuming $V_1,\ldots,V_{n-1}$ defined, find $V_n(v)\subset V_{n-1}\cap U_n$ such that $V_n\cap D$ is path-connected in $V_{n-1}\cap U_n$. Choose $d_i\in V_i\cap D$; according to the construction, there is for each $i=1,2,\ldots$ a path a_i from d_i to d_{i+1} such that $a_i(I)\subset U_i$. Define $a\colon I\to Z$ by

$$a(0)=v\,,\quad t=0\,,$$
$$a(t)=a_n[(n+1)(1-nt)]\,,\quad \frac{1}{n+1}<t\leqslant\frac{1}{n}\,,\quad n=1,2,\ldots$$

This is clearly continuous at $t=0$, because of the behaviour of the a_i, and provides a path from v to d_1 lying in U.

Proof of Theorem. According to 4.3, the set $D=p_{\#}(X^{S^k})$ is dense in $Z=Y^{S^k}$ and according to 4.2, the remaining requirement of the Lemma is satisfied, because $p_{\#}$ is continuous. Thus, Y^{S^k} is LC^0; and this implies, as is well-known, that Y is k-LC: given $y\in Y$ and any nbd $U(y)$, form the open covering $\mathfrak{U}=\{U,Y-y\}$ of Y, and let $c\colon S^k\to y$ be the constant map; since Y^{S^k} is LC^0, there is a refinement $\mathfrak{V}$ of $\mathfrak{U}$ such that $(c,\mathfrak{V})$ is path-connected in $(c,\mathfrak{U})$; so, if $V\in\mathfrak{V}$ is a set containing y, any $f\colon S^k\to V$ is nullhomotopic in U. Finally, Y must be first countable, since it can be embedded as a retract of Y^{S^k}.

Since for compact X, the c-topology in Y^X is metrizable whenever Y is metrizable, 4.4 gives

4.6. COROLLARY. *Let X be arbitrary and $p\colon X\to Y$ a continuous closed surjection with PC_X^n fibers. If Y is metrizable, then Y must be LC^n.*

In the special case that the fibers are PC_X^n because of 2.4, this result can be improved:

4.7. *Let X be paracompact and LC^{n+1}, let Y be metrizable, and let $p\colon X\to Y$ be a continuous closed surjection with n-connected LC^n fibers. Then Y is LC^{n+1}.*

Proof. Because the fibers are PC_X^n, it follows from 4.6 that Y is LC^n; we now show $Y^{S^{n+1}}$ is LC^0. This will follow from 4.5 by showing that for each $f\in Y^{S^{n+1}}$, each nbd $(f,\mathfrak{U})$ contains a nbd $(f,\mathfrak{V})$ such that $p_{\#}(X^{S^{n+1}})\cap(f,\mathfrak{V})$ is path-connected in $(f,\mathfrak{U})$.

For each $y\in Y$, let $U_y\in\mathfrak{U}$ be a set of the covering containing y. It follows easily from 2.3 that for each U_y there is a nbd $V_y\subset U_y$ of y such that any $h\colon S^{n+1}\to\widetilde{V}_y$ is homotopic in $\widetilde{U}_y$ is an $h'\colon S^{n+1}\to p^{-1}(y)$. Let $\mathfrak{V}=\{V_y|\ y\in Y\}$ and $\mathfrak{V}^*$ be a star-refinement.

Let Q denote the n-skeleton of S^{n+1} in some simplicial subdivision. Since the fibers are PC_X^n, there is, by 4.2, a nbd $(f|Q,\mathfrak{W})$ such that $p_{\#}^{-1}(f|Q,\mathfrak{W})$ is path-connected in $p_{\#}^{-1}(f|Q,\mathfrak{V}^*)$.

Now let $g,g'\colon S^{n+1}\to X$ be such that $p_{\#}g,p_{\#}g'\in(f,\mathfrak{W})$; then $g|Q\simeq g'|Q$ by a $\widetilde{\mathfrak{V}}^*$-homotopy H, so $H[(\sigma^{n+1}\times I)^{\cdot}]$ lies in some $\widetilde{V}_y\in\widetilde{\mathfrak{V}}$ for each closed $(n+1)$-simplex σ^{n+1} of S^{n+1}. Since the map $H|(\sigma^{n+1}\times I)^{\cdot}$ of an $(n+1)$-sphere into $\widetilde{V}_y$ is deformable over $\widetilde{U}_y$ into the fiber $p^{-1}(y)$, the map $p_{\#}H|(\sigma^{n+1}\times I)^{\cdot}$ is null homotopic over $U_y\in\mathfrak{U}$; extending $p_{\#}H$ over each $\sigma^{n+1}\times I$, in this manner, yields the required homotopy of $p_{\#}g$ to $p_{\#}g'$, and completes the proof.

5. Homotopy behaviour of p.

In this section, we consider the behaviour of p on the homotopy groups. If $p\colon X\to Y$ and $B\subset Y$, the map $p|p^{-1}(B)\colon p^{-1}(B)\to B$ is denoted by p^B.

5.1. THEOREM. *Let X be arbitrary, Y paracompact, and $p\colon X\to Y$ a continuous closed surjection with PC_X^n fibers. Then for each open* [8] *set $U\subset Y$, the induced homomorphism $p_*^U\colon\pi_q(\widetilde{U})\to\pi_q(U)$ is monic for $0\leqslant q\leqslant n$, and the induced homomorphism $p_*\colon\pi_q(X,\widetilde{U})\to\pi_q(Y,U)$ is monic for $1\leqslant q\leqslant n$.*

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[8] Recall that an open subset of a paracompact space may not itself be paracompact.

Proof. We prove the latter assertion, that for the former being similar. Let $a \in \pi_q(X, \tilde{U}, x_0)$ be represented by $g\colon (V^q, \dot{V}^q, v_0) \to (X, \tilde{U}, x_0)$ and assume $p_*(a) = 0$, so that there is a homotopy $H\colon V^q \times I \to Y$ such that $H|V^q \times 0 = pg$, $H(V^q \times 1) \subset U$ and $H(v, t) = H(v, 0)$ for $(v, t) \in \dot{V}^q \times I$. Define $\tilde{g}\colon V^q \times 0 \cup \dot{V}^q \times I \to X$ by $\tilde{g}(v, 0) = \tilde{g}(v, t) = g(v)$; then $p\tilde{g} = H|V^q \times 0 \cup \dot{V}^q \times I$ and, given the open covering $\mathfrak{U} = \{U, Y - H(V^q \times 1)\}$, there is by 3.1 an extension $\tilde{G}\colon V^q \times I \to X$ of $\tilde{g}$ such that $p\tilde{G} \in (H, \mathfrak{U})$; thus, $\tilde{G}(V^q \times 1) \subset \tilde{U}$ and therefore $a = 0$.

In particular, $p_*\colon \pi_i(X) \to \pi_i(Y)$ is monic for $0 \leqslant i \leqslant n$. However, if Y is dominated by a polytope (e.g., belongs to Milnor's [7] category $\mathfrak{W}$) then this can be improved:

5.2. Theorem. *Let X be arbitrary, Y paracompact, and $p\colon X \to Y$ a continuous closed surjection with PC_X^n fibers. If Y is dominated by a polytope, then $p_*\colon \pi_q(X) \to \pi_q(Y)$ is an isomorphism for $0 \leqslant q \leqslant n$, and epic for $q = n+1$.*

Proof. We need show only that p_* is epic. Choose base points $x_0 \in X$ and $y_0 = p(x_0)$ for the homotopy groups. Let P be a dominating polytope, and $\varkappa\colon Y \to P$, $g\colon P \to Y$ such that $g \circ \varkappa \simeq \mathrm{id}$. Let $a \in \pi_q(Y, y_0)$ be represented by $f\colon (S^q, s_0) \to (Y, y_0)$ and choose the covering $\mathfrak{U} = \{\varkappa^{-1}(\mathrm{St}\,p) \mid p \in P^0\}$ for Y. According to 4.3, there is, provided $q \leqslant n+1$, an $h\colon (S^q, s_0) \to (X, x_0)$ such that $ph \in (f, \mathfrak{U})$. Since $\varkappa f$ and $\varkappa ph$ are $\{\mathrm{St}\,p\}$-close, they are ([3], p. 215) also $\{\mathrm{St}\,p\}$-homotopic, and consequently homotopic rel s_0. Thus $g\varkappa f$ and $g\varkappa ph$ are homotopic and, since $g\varkappa \simeq 1$, we find f homotopic to ph. This completes the proof.

To have the p_*^U isomorphisms for every open $U \subset Y$, rather than for just $U = Y$, is a strong requirement, for we show

5.3. Theorem. *Let X be arbitrary, Y paracompact, and $p\colon X \to Y$ a continuous closed surjection with PC_X^n fibers. The following two statements are equivalent:*

(a) *Y is LC^n,*

(b) *$p_*^U\colon \pi_q(\tilde{U}) \approx \pi_q(U)$ for all open $U \subset Y$ and all $0 \leqslant q \leqslant n$.*

Proof. (a) $\Rightarrow$ (b). Let $a \in \pi_q(U, u_0)$ be represented by $f\colon (S^q, s_0) \to (U, u_0)$. Since Y is LC^n, then using the open covering $\mathfrak{U} = \{U, Y - f(S^q)\}$ of Y, there is, by 3.2, an open refinement $\mathfrak{V}$ such that $\mathfrak{V}$-close maps $(S^q, s) \to (U, u_0)$ are $\mathfrak{U}$-homotopic rel s. Since (4.3) $p_\#(X^{S^q})$ is dense in Y^{S^q}, there is a $g\colon (S^q, s) \to (\tilde{U}, x_0)$ with $pg \in (f, \mathfrak{V})$, consequently pg is homotopic to f over U.

(b) $\Rightarrow$ (a). Let $y_0 \in Y$, and let $F = p^{-1}(y_0)$. Because F is PC_X^n, given any nbd $U(y_0)$ there is a nbd $V(y_0) \subset U$ such that $\pi_q(\tilde{V} | \tilde{U}) = 0$ for $0 \leqslant q \leqslant n$. Letting $i\colon \tilde{V} \to \tilde{U}$ and $j\colon V \to U$ be the inclusion maps, we have the commutative diagram

$$\begin{array}{ccc} \pi_q(\tilde{V}) & \xrightarrow{i_*} & \pi_q(\tilde{U}) \\ {\scriptstyle p_*^V}\downarrow & & \downarrow{\scriptstyle p_*^U} \\ \pi_q(V) & \xrightarrow[j_*]{} & \pi_q(U)\,. \end{array}$$

For $0 \leqslant q \leqslant n$, i_* is the zero homomorphism so, since p_*^U, p_*^V are isomorphisms, j_* is the zero homomorphism. Thus (⁹), Y is LC^n at y_0.

In the case that Y is metrizable, then (5.3 and 4.6) we have

5.4. Theorem. *Let X be arbitrary, Y metrizable, and $p\colon X \to Y$ a continuous closed surjection with PC_X^n fibers. Then Y is LC^n and therefore $p_*^U\colon \pi_q(\tilde{U}) \approx \pi_q(U)$ for every open set $U \subset Y$ and $0 \leqslant q \leqslant n$.*

This remains true if Y is paracompact and Y^{S^i}, $0 \leqslant i \leqslant n$, are first countable.

6. Applications.

We give here only some immediate applications of the main results.

The following generalization of the result in [8], [9], has also been obtained in [5].

6.1. Theorem. *Let X be a paracompact LC^n space, and $p\colon X \to Y$ a continuous closed surjection in which each fiber is LC^{n-1} and $(n-1)$-connected. If Y is metrizable, then Y is LC^n and $p_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $0 \leqslant i \leqslant n-1$ and epic for $i = n$.*

Proof. That Y is LC^n follows from 4.7. Since the fibers are PC_X^{n-1} then because of 5.3 we need prove only that $p_*\colon \pi_n(X) \to \pi_n(Y)$ is epic. According to 4.7, the space Y^{S^n} is LC^0 so that the path components of Y^{S^n} are open sets; the dense set $p_\#(X^{S^n})$ therefore meets each path-component, so p_* is epic.

If X is a metric space, and $p\colon X \to Y$ is a continuous closed surjection, then by Michael's theorem ([4], p. 165) the space Y is paracompact and, by the Stone–Hanai theorem ([4], p. 235) Y is metrizable whenever all the fibers are compact. Thus, if X is an ANR and $p\colon X \to Y$ is a continuous closed surjection with AR fibers, then it follows from 5.2, 5.4 that

(a) If Y is dominated by a polytope, then $p_*\colon \pi_i(X) \approx \pi_i(Y)$ for all $i \geqslant 0$ so that p is in fact a homotopy equivalence,

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(⁹) Observe that, by using the 5-Lemma, it follows immediately from 5.3(b) that also $p_*\colon \pi_q(\tilde{U}, \tilde{V}) \approx \pi_q(U, V)$ for $1 \leqslant q \leqslant n$ and all pairs $V \subset U$ of open sets in Y.

and

(b) If the fibers are compact, then Y is LC^{∞} and p is a weak homotopy equivalence; moreover, if $\dim Y < \infty$, then Y is an ANR and p a homotopy equivalence. These results contain those in ([1], p. 127).

We also obtain (compare [10], p. 487)

6.2. *Let X be paracompact and $A \subset X$ a closed PC_X^n subset. Let $p\colon X \to X/A$ be the projection. If X/A is dominated by a polytope, then $p_*\colon \pi_i(X) \to \pi_i(X/A)$ is an isomorphism for $0 \leqslant i \leqslant n$, and is epic for $i = n+1$.*

Proof. Since p is a continuous closed surjection, Michael's theorem shows X/A is paracompact so 5.2 is applicable.

Because of 2.2, it follows that under the hypotheses of 6.2, we have $\pi_i(X, A) \approx \pi_i(X/A) \oplus \pi_{i-1}(A)$ for $2 \leqslant i \leqslant n$.

We also determine some conditions under which each fiber in a Serre fibration is PC_E^n.

6.3. *Let (E, p, B) be a Serre fibration, where E is compact and B is dominated by a polytope. If each fiber F is PC_E^n, then each fiber is n-connected.*

Proof. Since $\pi_i(F|E) = 0$ for $0 \leqslant i \leqslant n$ (cf. 2.2) the homotopy sequence of (E, p, B) decomposes into short exact sequences

$$0 \to \pi_i(E) \xrightarrow{p_*} \pi_i(B) \to \pi_{i-1}(F) \to 0 \qquad 0 \leqslant i \leqslant n$$

and a long exact sequence $\ldots \to \pi_{n+1}(E) \xrightarrow{p_*} \pi_{n+1}(B) \to \pi_n(F) \to 0$. Because E is compact, p is a closed map so, by 5.2, p_* is an isomorphism for $0 \leqslant i \leqslant n$ and epic for $i = n+1$; from the exact sequences we find $\pi_i(F) = 0$ for $0 \leqslant i \leqslant n$.

It is trivial to verify that, in a Serre fibration (E, p, B), if B is LC^n and if each fiber F is n-connected, then each fiber F is PC_E^n. Thus,

6.4. *Let E be compact, B a polytope and (E, p, B) a Serre fibration. Then every fiber is PC_E^n if and only if every fiber is n-connected.*

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