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# DECOMPOSITIONS THAT DESTROY SIMPLE CONNECTIVITY

BY

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We shall be concerned with a monotone decomposition of $R^3$ with only one nondegenerate decomposition element $X$. We use $g$ to denote the decomposition map and $g(R^3)$ the decomposition space. Also, $D$ denotes a disk. To determine if $g(R^3)$ is simply connected we shall be concerned with whether maps of Bd $D$ into $g(R^3)$ can be extended to $D$.

At the Summer Institute on Set Theoretic Topology at Wisconsin in 1955 I gave a talk entitled "What topology is here to stay" in which I envisioned decompositions of $R^3$ as a very viable area for research. I mentioned R.L. Moore's monotone decomposition theorem [3] for $S^2$ which states that if $G$ is a nondegenerate upper semicontinuous decomposition of $S^2$ each of whose elements is a continuum that does not separate $S^2$, then the decomposition space is $S^2$. I pointed out that the theorem was false if one replaced $S^2$ by $S^3$ and gave as an example the decomposition whose only nondegenerate element is a circle. The earlier version of the Summary of Lectures and Seminars [1] reported on page 26 that the reason I gave that the decomposition space differed from $S^3$ was that it *is not simply connected*. The second printing of [1] made the correction by replacing the *is not simply connected* part of the statement by *does not remain simply connected* on the removal of some point. It was also claimed there and in [2] that the decomposition space of $S^3$ (or $R^3$) whose only nondegenerate element is a solenoid is not simply connected. When I was assembling copies of my publications it was called to my attention that a proof of this claim had not been published. It is the purpose of this paper to fill that gap. Other claims were made in [2] about the simple connectivity of other monotone decompositions (perhaps with many nondegenerate elements) of $R^3$, but we shall not treat them in this paper.

Richard Skora read an early draft of this paper and made valuable suggestions for improving some proofs.

## 1. $X$ is a standard solenoid

In this case $X$ is the intersection of smooth unknotted tori $T_1, T_2, \ldots$ where $T_{i+1}$ winds around $T_i$ smoothly more than once, the meridional cross sections

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of $T_{i+1}$ are round planar disks that lie in those of $T_i$, and the diameters of the meridional cross sections of the $T_i$'s converge to 0 as $i$ increases without limit. Sometimes the restriction of "more than once" is omitted and a circle is permitted to be a solenoid—but we shall not do that in this paper.

THEOREM 1. *If $X$ is a standardly embedded solenoid, $g(R^3)$ is neither simply connected nor locally simply connected.*

*Proof.* If we seek a map $f$ of Bd $D$ into $g(R^3)$ that cannot be extended to $D$, we should seek one such that $g(X) \in f(\text{Bd } D)$ because if $g(X) \notin f(\text{Bd } D)$ there is an extension of $g^{-1}f$ on Bd $D$ to take $D$ into $R^3$. This extension followed by $g$ would extend $f$ to map $D$ into $g(R^3)$.

Let $pq$ be an arc in a meridional cross section of $T_1$ that intersects $X$ only in its end points where these end points belong to different arc components of $X$. We show that $g(R^3)$ is not simply connected by showing that a homeomorphism $f$ of Bd $D$ onto $g(pq)$ cannot be extended to map $D$ into $g(R^3)$. Since for each open subset $U$ of $g(R^3)$ containing $g(X)$ there is a $pq$ with $g(pq)$ in $U$, this will also show that $g(R^3)$ is not locally simply connected. See Figure 1.

Assume $f$ is a homeomorphism of Bd $D$ onto $g(pq)$ and that $f$ can be extended to a map $F$ of $D$ into $g(R^3)$. It would be nice if $F^{-1}(g(X))$ were 0-dimensional, so we adjust $F$ to get a new map $F_2$ where $F_2^{-1}(g(X))$ is 0-dimensional. First, let $F_1$ be a map of $D$ into $g(R^3)$ such that $F_1 = F$ on the component of $D - F^{-1}(g(X))$ intersecting Bd $D$ and $F_1$ takes the rest of $D$ to $g(X)$. Next we let $k$ be a map of $D$ onto itself that is the identity on Bd $D$ and whose point inverses are the components of $F_1^{-1}(g(X))$ and points of $D - F_1^{-1}(g(X))$. Moore's decomposition theorem [3] mentioned earlier is used to get $k$. Then $F_2 = F_1k^{-1}$. For simplicity we suppose $F = F_2$.

Let $a_0b_0$ be a spanning arc of $D$ such that $F(a_0b_0)$ misses $g(X)$ and the subdisk $D_0$ of $D$ bounded by the union of $a_0b_0$ and the subarc of Bd $D$ from $a_0$ to $b_0$ through $f^{-1}(g(X))$ lies in $F^{-1}(g(\text{Int } T_1))$. Since $g^{-1}(F(a_0))$ and $g^{-1}(F(b_0))$ lie in the same meridional cross section of $T_1$, we can speak of the number of times that $g^{-1}F(a_0b_0)$ winds around $T_1$.

For some large $r$ let $a_rb_r$ be a spanning subarc of $D_0$ such that $a_r$ lies on Bd$D$ between $a_0$ and $f^{-1}g(X)$, $b_r$ lies on Bd $D$ between $b_0$ and $f^{-1}g(X)$, $F(a_rb_r)$ misses $g(X)$, and $F(a_rb_r)$ lies in $g(T_r)$. Let $D_r$ be the subdisk of $D_0$ bounded by union of $a_0b_0$, $a_rb_r$ and two subarcs of Bd $D$. See Figure 1.

Since $p$ and $q$ belong to different arc components of $X$, for large $r$, $g^{-1}F(a_rb_r)$ winds around $T_1$ many times—even more than $g^{-1}F(a_0b_0)$ does. Let $y(r)$ be the number of times that $g^{-1}(F(\text{Bd } D_r))$ winds around $T_1$. Suppose $r$ is so large that $y(r) > 0$.

Let $z(s)$ be the number of times that $T_s$ winds around $T_1$. We suppose $s$ is so large that $z(s) > y(r)$ and $g^{-1}F(\text{Bd } D_r)$ misses Bd $T_s$. We suppose that on $D_r$ near $F^{-1}(g(\text{Bd } T_s))$, $F$ has enough general position so that $D \cap F^{-1}g(\text{Bd } T_s)$ is the union of a finite number of mutually disjoint simply closed

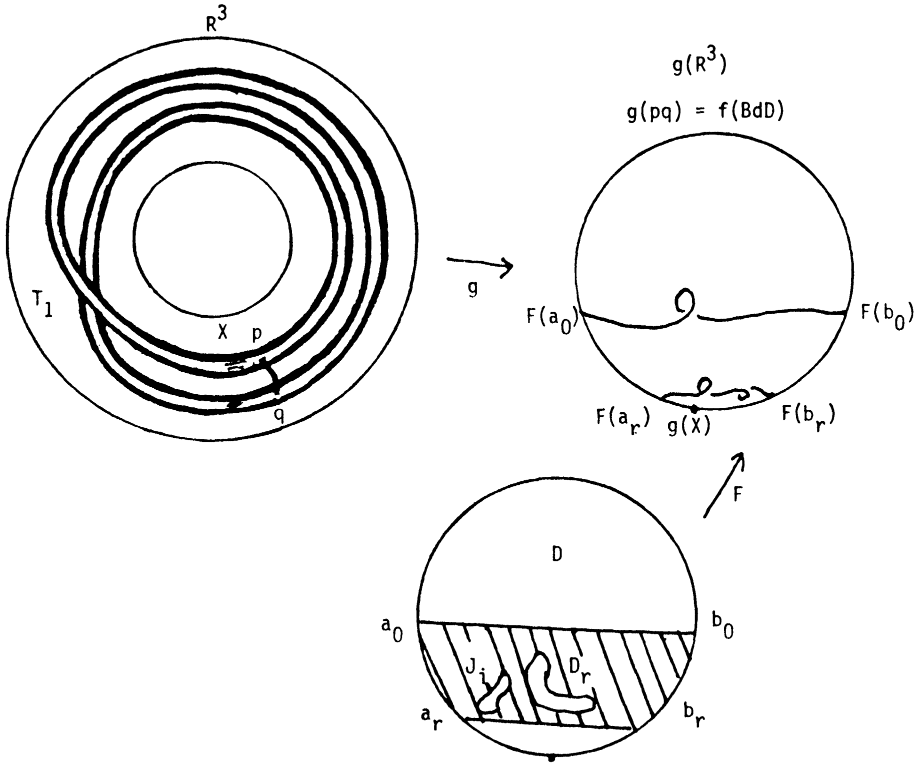


FIG. 1

curves $J_1, J_2, \ldots, J_n$. Since each $g^{-1}F(J_i)$ lies on Bd $T_s$, it winds around $T_1$ some integral multiple of $z(s)$.

Let $E_r$ be the finitely holed $D_r$ obtained by deleting from $D_r$ the interiors of the subdisks of $D_r$ bounded by the $J_i$'s. We now come to the contradiction caused by the assumption tha $f$ on Bd $D$ could be extended to $F$ on $D$. The boundary of each of the holes of $E_r$ winds around $T_1$ some integral multiple of $z(s)$, but $y(r)$ is not an integral multiple of $z(s)$.

## 2. $X$ is an embedded solenoid

The complement of an embedded solenoid may be quite different from the complement of a standardly embedded solenoid. We no longer can speak of tori about the embedded solenoid. However, we still find that the decomposition space is not simply connected.

THEOREM 2. *If $X$ is an embedded solenoid, $g(R^3)$ is neither simply connected nor locally simply connected.*

*Proof.* We use $X'$ to denote the standardly embedded solenoid of Theorem 1, $p'q'$ the arc called $pq$ there, and $g'$ the decomposition map called $g$ there. Let $\beta$ be a homeomorphism of $X$ onto $X'$ and $pq$ an arc from $\beta^{-1}(p')$ to $\beta^{-1}(q')$ in $R^3$ that intersects $X$ only at $p$ and $q$. This is possible since the dimension of $X$ is 1. Extend $\beta$ to a map of $R^3$ onto itself that takes $pq$ homeomorphically to $p'q'$. For convenience, call the extension $\beta$. It is a map rather than a homeomorphism. We finish the proof of Theorem 2 by showing that $g(pq)$ cannot be shrunk to a point in $g(R^3)$.

Let $f$ be a homeomorphism of Bd $D$ onto $g(pq)$. Assume that $f$ can be extended to map $F$ taking $D$ into $g(R^3)$. This leads to the contradiction that a homeomorphism of Bd $D$ onto $g'(p'q')$ can be extended to a map $g'\beta g^{-1}F$ of $D$ into $g'(R^3)$. See Figure 2. The proof of Theorem 1 showed that no homeomorphism of Bd $D$ onto $g'(p'q')$ could be extended to map $D$ into $g'(R^3)$.

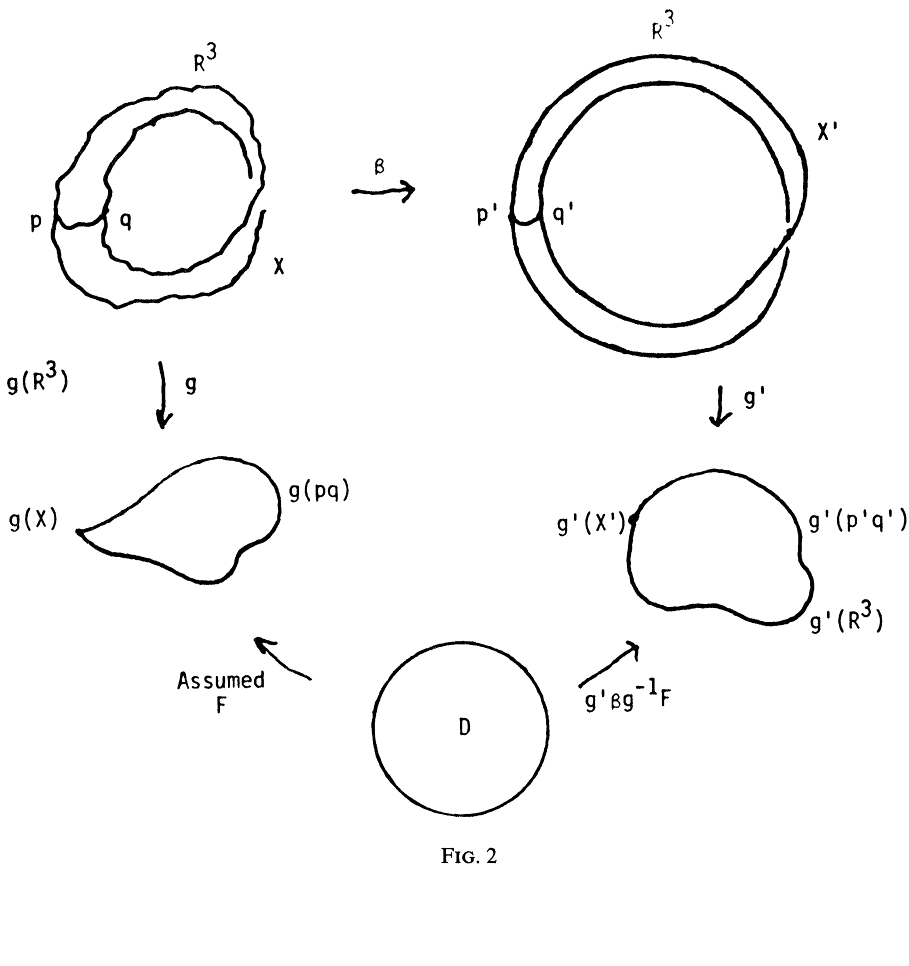


FIG. 2

Since $pq$ could be picked close to $X$, this also shows that $g(X)$ is not locally simply connected.

## 3. $X$ is unlike-a-solenoid

We say that a disk $D$ can be converted to a disk with finitely many holes $E$ if there is a finite collection of mutually disjoint disks in Int $D$ and $E$ is obtained from $D$ by removing the interiors of these subdisks. These interiors are called holes in $D$ and $E$ is called a *finitely holed* $D$. We call $D$ a finitely holed $D$ even if there are no holes and $D = E$.

Let $N_i$ be the $1/i$-neighborhood of $X$ in $R^3$—that is, the set of points of $R^3$ whose distance from $X$ is less than $1/i$.

If $f(\text{Bd } D) \subset N_i - X$, we say that *$f$ can be pulled in $N_i - X$ arbitrarily close to $X$* if for arbitrary large $s$, $f$ can be extended to take a finitely holed $D$ into $N_i - X$ so that the boundary of each hole is sent into $N_s$. Note that $s$ is picked before the extension. If a different $s$ had been chosen, we might have needed a different extension. (If $f$ can be extended to map $D$ into $N_i$, we could have picked an extension independent of $s$. If $f$ can be extended to map $D$ into $N_i - X$, then technically the definition says that $f$ can be pulled arbitrarily close to $X$ even though $f(D)$ misses $X$. This is because we call $D$ a finitely holed $D$.) We say that $X$ *is unlike-a-solenoid* if for each $N_i$ there is an $N_{r(i)}$ such that each map $f$ of Bd $D$ into $N_{r(i)} - X$ can be pulled in $N_i - X$ arbitrarily close to $X$.

THEOREM 3. *If $X$ is unlike-a-solenoid, then $g(R^3)$ is simply connected and locally simply connected.*

*Proof.* We first show that $g(R^3)$ is locally simply connected at $g(X)$. We show that if $U$ is a neighborhood in $g(R^3)$ of $g(X)$, there is a neighborhood $V$ of $g(X)$ such that each map $f$ of Bd $D$ into $V$ can be extended to take $D$ into $U$. We use $f$ as a map of Bd $D$ into $V$ and $g^{-1}f$ to send Bd $D$ into $R^3$.

To show that $g(R^3)$ is locally simply connected, without loss of generality we pick $U$ to be $g(N_j)$ and $V$ to be $g(N_{r(j)})$ where $r(j)$ is an integer such that any map of Bd $D$ into $N_{r(j)} - X$ can be pulled in $N_j - X$ arbitrarily close to $X$ on a finitely holed $D$.

We consider the sequence $n_1, n_2, \dots$ where $n_1 = j$, and $n_{i+1} = r(n_i)$. Although the $r$'s are defined for maps of Bd $D$ into $R^3 - X$, we realize that in the case of the $f$ of Bd $D$ into $V = g(N_{r(j)})$ we wish to extend it to a map from $D$ into $U = g(N_j)$ and this $f(\text{Bd } D)$ may contain $g(X)$.

It may be that $f^{-1}(g(X))$ has several components. We wish to avoid this. With that purpose in mind we suppose $D$ is a round planar disk and let $C$ be the convex hull of $f^{-1}(g(X))$ and partially extend $f$ to send $C$ to $g(X)$. For convenience we call the extension $f$. Now $f$ is defined except on a collection

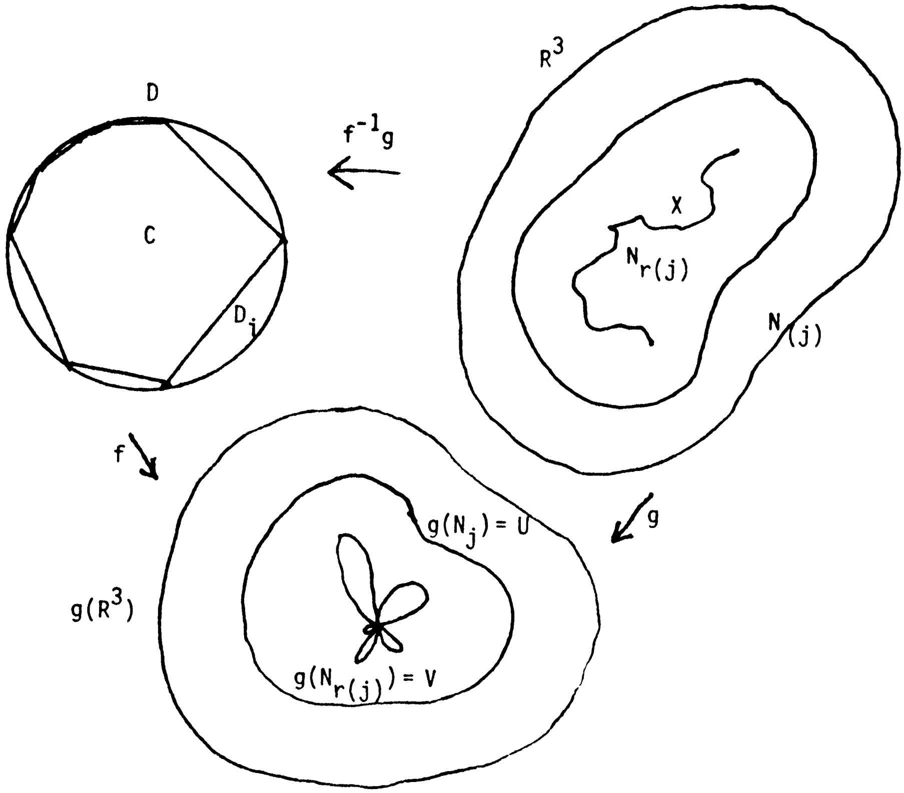


FIG. 3

(possibly infinite) of open disks. We call these disks $D_i$'s. For each of these disks, $f$ sends the straight subarc of its boundary to $g(X)$ and the open curved part into $g(R^3 - X)$. We have simplified the situation so that $f^{-1}g(X)$ intersects each Bd $D_i$ in a connected set. See Figure 3.

Let $D_i$ be one of the subdisks on whose interior $f$ has not been defined. Each Bd $D_i \subset g(N_{n_2})$ and for all but a finite number of these $D_i$'s $f(\text{Bd } D_i) \subset g(N_{n_3})$. For this finite number where $f(\text{Bd } D_i) \not\subset g(N_{n_3})$, we extend $f$ to a part of their interiors.

Let $ab$ be a spanning arc of $D_i$ that cuts $D_i$ into two subdisks $D_i'$, $D_i''$ where $f(\text{Bd } D_i' - ab)$ misses $g(X)$ and the closure of $f(\text{Bd } D_i'' - ab)$ lies in $g(N_3)$. Extend $f$ to take $ab$ into $g(N_{n_3} - X)$. Extend $f$ further on a finitely holed $D_i'$ to take the finitely holed $D_i'$ into $g(N_{n_1} - X)$ where the boundary of the holes go into $g(N_{n_3})$. To get the extension we use the hypothesis that $X$ is unlike-a-solenoid.

We now find that $f$ is defined except on open disks whose boundaries are sent by $f$ into $g(N_{n_3})$. In fact on all but a finite number of these disks, $f$ sends their boundaries into $g(N_{n_4})$. We extend $f$ into a part of the interiors of this finite collection so that now $f$ is defined except on open disks whose boundaries are sent into $g(N_{n_4})$. Using the hypothesis that $X$ is unlike-a-solenoid we pick the extension on the new part to take this new part into $g(N_{n_2})$.

The extension is extended a countable number of times and finally we define the extension to take the remaining part of $D$ to $g(X)$. We have now shown that $g(R^3)$ is locally simply connected. That it is simply connected follows from the following theorem.

THEOREM 4. *If $g(R^3)$ is locally simply connected, it is simply connected.*

*Proof.* Let $f$ be a map of Bd $D$ into $g(R^3)$. We show that $f$ is simply connected by showing that $f$ can be extended to map $D$ into $g(R^3)$.

It follows from the local simple connectivity of $g(R^3)$ that there is a neighborhood $U$ of $g(X)$ such that each map of Bd $D$ into $U$ can be extended to map $D$ into $g(R^3)$. Suppose $D$ is a round disk and $A_1, A_2, \dots, A_n$ are the components of Bd $D - f^{-1}g(X)$ that are not sent into $U$ by $f$. Let $B_i$ be an open arc on $A_i$ such that each $f(A_i - B_i) \subset U$. Let $C_i$ be a straight arc in $D$ joining the two components of $A_i - B_i$.

Extend $f$ to $C_i$ so that the extension takes $C_i$ into $U - g(X)$. Call the extension $f$. On the subdisk $D_i$ of $D$ bounded by $C_i$ and a part of $A_i$, extend $g^{-1}f$ to take each $D_i$ into $R^3$ and follow this extension by $g$ to extend $f$ to take $D_i$ into $g(R^3)$. The boundary of the remaining part of $D$ is sent by $f$ into $U$ so the local connectivity of $g(R^3)$ shows that $f$ can be extended to the rest of $D$. This shows that $g(R^3)$ is simply connected.

## 4. $X$ is solenoid-like

We say that $X$ is solenoid-like if there is a neighborhood $N$ of $X$ in $R^3$ such that for any neighborhood $N'$ of $X$ there is a map $f$ of Bd $D$ into $N' - X$ which cannot be pulled in $N - X$ on a finitely holed $D$ arbitrarily close to $X$. One might note if $X$ is solenoid-like, then it is untrue that $X$ is unlike-a-solenoid.

THEOREM 5. *If $X$ is solenoid-like, $g(R^3)$ is not locally simply connected.*

*Proof.* We show that for some neighborhood $U = g(N)$ of $g(X)$, and each smaller neighborhood $V = g(N')$ of $g(X)$ there is a map of Bd $D$ into $V$ that cannot be extended to map $D$ into $U$. Here we use $N, N', f$ as in definition of solenoid-like and use $gf$ for the map of Bd $D$ into $V - g(X)$ that cannot be extended to map $D$ into $U$. If $gf$ could be extended by $F$ to send $D$ into $U$, $g^{-1}F$ would show that $f$ can be pulled in $N - X$ on a finitely holed $D$ arbitrarily close to $X$.

*Question* Recall that $X$ is a continuum in $R^3$ and $g(R^3)$ is the decomposition space whose only nondegenerate point inverse is $X$. Is $g(R^3)$ locally simply connected if it is simply connected? If $g(R^3)$ is not locally simply connected, could it be simply connected?

## 5. Necessary and sufficient conditions

Theorems 3 and 5 provide a necessary and sufficient condition that $g(R^3)$ is not locally simply connected. However the condition is dependent on the embedding of $X$ and does not say that if $X'$ is homeomorphic to $X$ then $g(R^3) = R^3/X$ is locally simply connected if and only if $R^3/X'$ is.

THEOREM 6. *A necessary and sufficient condition that* $g(R^3)$ *not be locally simply connected is that* $X$ *be solenoid-like.*

*Proof.* The sufficiency is provided by Theorem 5 and the necessity by Theorem 3.

THEOREM 7. *If* $X$ *and* $X'$ *are homeomorphic continua in* $R^3$ *and* $\dim X = 1$, *then* $g(R^3) = R^3/X$ *fails to be locally simply connected if and only if* $g'(R^3) = R^3/X'$ *does.*

*Proof.* The proof of Theorem 7 is modelled after that of Theorem 2.

Suppose $g'(R^3)$ is not locally simply connected. Then it is solenoid-like and there are a neighborhood $N'$ of $X'$ and a sequence of mutually disjoint simple closed curves $J_1', J_2', \ldots$ in $N' - X'$ such that $J_i'$ lies in the $1/i$ neighborhood of $X'$ and a map $f_i'$ of Bd $D$ onto $J_i'$ that cannot be pulled in $N' - X'$ arbitrarily close to $X'$ on a finitely-holed $D$.

Let $\beta$ be a homeomorphism of $X$ onto $X'$. We now pick a sequence of simple closed curves $J_1, J_2, \ldots$ in $R^3 - X$ so that $\beta$ can be extended to a homeomorphism taking $X \cup J_1 \cup J_2 \cup \ldots$ onto $X' \cup J_1' \cup J_2' \cup \ldots$. We assume $J_1, J_2, \ldots, J_{i-1}$ have been found with $\beta$ extended to them and describe $J_i$ and $\beta$ on it.

Express $J_i'$ as the union of arcs $a_1'a_2', a_2'a_3', \ldots, a_n'a_{n+1}'$ each of diameter less than $1/i$. Let $b_j'$ be a point of $X'$ in the $1/i$ neighborhood of $a_j'$. Note that the distance between two adjacent $b_j'$'s is less than $3/i$. Let $a_j$ be a point of

$$R^3 - (X \cup J_1 \cup \cdots \cup J_{i-1})$$

in the $1/i$ neighborhood of $\beta^{-1}(a'_j)$. Note that distance between two adjacent $a_j$'s is less than $\varepsilon + 2/i$ where $\varepsilon$ is a positive number such that the image under $\beta^{-1}$ of any $3/i$-subset of $X'$ has diameter less than $\varepsilon$. Hence adjacent $a_j$'s are close if $i$ is large. We suppose the $a_j$'s are distinct and let $J_i$ be a simple closed curve in

$$R^3 - (X \cup J_1 \cup J_2 \cup \cdots \cup J_{i-1})$$

which is the union of arcs $a_1a_2, a_2a_3, \ldots, a_na_{n+1}$ where $a_ja_{j+1}$ lies in the $1/i$ neighborhood of the straight line interval from $a_j$ to $a_{j+1}$. It is in showing that there are such $a_ja_{j+1}$'s in $R^3 - (X \cup J_1 \cup J_2 \cup \cdots \cup J_{i-1})$ that we use the fact that dim $X = 1$. The homeomorphism $\beta$ is extended to $J_i$ so that $\beta(a_j) = a'_j$.

Suppose all $J_i$'s are defined. Call the extended homeomorphism $\beta$. Now extend $\beta$ to a map taking $R^3$ onto $R^3$ and call this extension $\beta$ also. This final extension need not be a homeomorphism. Let $f_i$ be a homeomorphism of Bd $D$ onto $J_i$ such that $\beta f_i = f'_i$.

Assume that $g(R^3) = R^3/X$ is locally simply connected. We prove the theorem by showing that this assumption is false. Local simple connectivity of $g(R^3)$ implies that for a large $r$, $gf_r$ on Bd $D$ can be extended to a map $F_r$ taking $D$ into $g\beta^{-1}(N')$. Let $C$ be the set of points of $D$ that are carried by $\beta g^{-1}F_r$ onto $X'$ or points of $X'$. Change $D$ to $E$, a finitely holed $D$ in $D - C$, so that the boundaries of the holes of $E$ are very close to $C$. Although $\beta g^{-1}F_r$ need not be a map on $D$, it is one on $E$. Also $\beta g^{-1}F_r = f'_r$ on Bd $D$. Hence $\beta g^{-1}F_r$ on $E$ shows that $f'_r$ on Bd $D$ can be pulled in $N' - X'$ arbitrarily close to $X$ on a finitely holed $D$. This contradicts the selection of $J'_r$ and $f'_r$.

## References


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